Infinitesimal Deformations
of
Singularities

A thesis presented

by

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to

The Department of Mathematics

in partial fulfillment of the requirements

for the degree of

Doctor of Philosophy

in the subject of

Mathematics

Harvard University

Cambridge, Massachusetts

June 1964

Introduction

Let A be an artin local ring with residue field k, and let Y be a flat scheme over Spec A. We may regard Y/Spec A as an (infinitesmal) deformation of the closed fibre $X = Y \times_{\text{Spec } A} \text{Spec } k$, with Spec A as the parameter space of the deformation. If Z/Spec A is another such deformation of X, we say that Y and Z are isomorphic (as deformations of X) if there exists an isomorphism $Y \xrightarrow{\sim} Z$ over Spec A which leaves the closed fibre X fixed. The problem which we wish to consider is the following: starting from a fixed scheme X over an algebraically closed field k, find all deformations Y/Spec A of X, for variable A; in other words, investigate the functor

$$A \rightsquigarrow \text{isomorphism classes of deformations}$$

$$Y/\text{Spec } A \text{ of } X$$

as A varies in the category $\underline{C}$ of artin local k algebras. In §3.3, we show that if X is proper over k, then there exists a complete local k algebra $R = \text{Lim } R/\underline{m}^n$ (with $R/\underline{m}^n \in \underline{C}$), and a sequence of deformations $X_n/\text{Spec}(R/\underline{m}^n)$ such that the formal R prescheme $\mathfrak{X} = \varinjlim X_n$ is the "generic deformation" of X; that is every deformation Y/Spec A ($A \in \underline{C}$) may be obtained from some homomorphism $R \to A$ by setting $Y = \mathfrak{X} \times_{\text{Spec } R} \text{Spec } A$, and the homomorphism $R \to A$ is uniquely determined by the isomorphism class of Y/Spec A if the square of the maximal ideal in A is zero.

The investigation of deformations composes essentially into two parts.

(a) Classify the linear deformations of X (the case where $A = k[\varepsilon]/\varepsilon^2$ is the algebra of dual numbers.)

(b) Given an arbitrary deformation Y/Spec A and a surjection $B \rightarrow A$ in $\underline{C}$, find the obstruction to extending Y to some deformation Z/Spec B (where we may assume that $A = B/J$ with $\underline{m}_A J = (0)$).

In the case of nonsingular X, Grothendieck has shown, in SGA III, that the set of linear deformations is a vector space isomorphic to $H^1(X, T)$ and that the obstruction in (b) lies in $H^2(X, T) \underset{k}{\times} J$, where T is the tangent sheaf to X. The essential point in this (non-singular) case is that one knows a priori that problems (a), (b) above are locally trivial. (In other words, if $x \in X$, then given any two deformations Y, Z of X, there exists an open neighborhood U of $x \in X$ such that $Y|U \simeq Z|U$, and in the same way, the obstruction in (b) vanishes locally.) To attack the general (singular) case, we use the cotangent complex developed in [6] by Stephen Lichtenbaum and the author. The cotangent complex furnishes coherent sheaves on X (by taking cohomology) which allows us to "count" the local deformations, and find the local obstructions to deforming X.

The method given here for constructing the generic deformation is simpler than that outlined by Grothendieck in [4]. It is a direct generalization of the technique used by Mumford to construct the "formal variety of moduli" for abelian varieties [7].

I should like to thank all those who have helped and encouraged me on this thesis, especially Stephen Lichtenbaum, David Mumford, Oscar Zariski, and, most of all my thesis advisor, John Tate.

Conventions

All rings are commutative, with identity. Ring homomorphisms preserve the identity, and all modules are assumed unitary.

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§1. The Cotangent Complex.

1. 1. Definition of the Cotangent Complex $L_{B/A}$.

Let $A \to B$ be a ring homomorphism, and write $B = P/I$ where P is a polynomial ring over A (finitely generated or not). Write

$$0 \to R \xrightarrow{i} F \xrightarrow{j} I \to 0 \qquad \text{(exact)}$$

where F is a free P module. Define

$$\lambda : \Lambda^2 F \to R$$

By $\lambda(x \wedge y) = jx \cdot y - jy \cdot x$, and let $R_0 = \text{image } \lambda$. Then we define a three term complex $L_{B/A}$ (depending on P and F) of B modules as follows: $L^0_{B/A} = \Omega_{P/A} \otimes_P B$ (where $\Omega_{P/A}$ is the module of Kähler differentials), $L^1_{B/A} = F \otimes_P B$, $L^2_{B/A} = R/R_0$. Thus

$$L_{B/A} : 0 \to R/R_0 \xrightarrow{d_2} F \otimes_P B \xrightarrow{d_1} \Omega_{P/A} \otimes_P B \to 0$$

where d_1, d_2 are defined as follows.

We have the (B linear) homomorphism $d : I/I^2 \to \Omega_{P/A} \otimes_A B$ induced from the A linear homomorphism $f \rightsquigarrow df$ of I into $\Omega_{P/A}$. d_1 is the composition

$$F \otimes_P B \xrightarrow{j \otimes 1} I/I^2 \xrightarrow{d} \Omega_{P/A} \otimes_P B .$$

d_2 is induced from i.

In [6] the following are proved:

(1) If

$$\begin{tikzcd} B \arrow[r] & B' \\ A \arrow[u] \arrow[r] & A' \arrow[u] \end{tikzcd}$$

is a commutative diagram of ring homomorphisms, then for any choice of P, R, P', R' we can associate a homomorphism

$$L_{B/A} \longrightarrow L_{B'/A'}$$

unique up to homotopy.

(2) $L_{B/A}$ commutes with flat base extension; i.e. if $A \longrightarrow A'$ is flat and $B' = B \otimes_A A'$, then $L_{B/A} \otimes_A A' \longrightarrow L_{B'/A'}$ is an isomorphism for suitable choice of $L_{B/A}$.

(3) $L_{B/A}$ commutes with localization on B; if S is a multiplicative system in B, then the map $(L_{B/A})S^{-1} \longrightarrow L_{BS^{-1}/A}$ is a homotopy equivalence.

(4) If $A \rightarrow B \rightarrow C$ is a sequence of ring homomorphisms, then for suitable choice of the cotangent complexes we have a sequence

$$0 \longrightarrow L_{B/A} \otimes_B C \longrightarrow L_{C/A} \longrightarrow L_{C/B} \longrightarrow 0$$

of complexes of C modules, which is <u>exact</u>, except that $L^2_{B/A} \otimes C \rightarrow L^2_{C/A}$ need not be injective.

(5) If B is <u>flat</u> over A, $A' \rightarrow A$ a ring homomorphism and $B' = B \otimes_A A'$, then

$$L_{B/A} \otimes_B B' \longrightarrow L_{B'/A'}$$

is a bijection, for suitable choice of $L_{B'/A'}$.

It follows from (1) that the homology and cohomology of $L_{B/A}$ are well defined. If for any B module M we put

$$T^i(B/A, M) = H^i(\mathrm{Hom}_B(L_{B/A}, M))$$

$$T_i(B/A, M) = H_i(L_{B/A} \otimes_B M)$$

For $i = 0, 1, 2$, then we get a nine term exact sequence in homology or cohomology. For the cohomology, e.g., this looks like

$$\begin{aligned} & 0 \longrightarrow T^0(C/B, M) \longrightarrow T^0(C/A, M) \longrightarrow T^0(B/A, M) \\ (6) \qquad & \longrightarrow T^1(C/B, M) \longrightarrow T^1(C/A, M) \longrightarrow T^1(B/A, M) \\ & \longrightarrow T^2(C/B, M) \longrightarrow T^2(C/A, M) \longrightarrow T^2(B/A, M). \end{aligned}$$

where $A \longrightarrow B \longrightarrow C$ is a sequence of ring homomorphisms and M is a C module. One has also, for fixed B/A and any exact sequence $0 \longrightarrow M' \longrightarrow M \longrightarrow M'' \longrightarrow 0$ of B modules, a corresponding nine term exact sequence in homology or cohomology.

1.2. Flatness.

We shall give here two criteria for the flatness of a "deformation". Recall that if A is a ring and M is an A module, we say that M is flat over A if $\mathrm{Tor}_1^A(M, N) = (0)$ for any A module N. From S.G.A.IV, Cor. 5.5, we have the following:

Proposition 1. Let A be a ring, J a nilpotent ideal in A, and M an A module. Then the following are equivalent:

(i) M/JM is flat over A/J, and $\operatorname{Tor}_1^A(M, A/J) = (0)$.

(ii) M is flat over A.

Corollary. Let A be a ring, J a nilpotent ideal in A, P a flat A algebra, I an ideal in P, and put $B = P/I$. Then the following are equivalent:

(1) B/JB is flat over A/J and $I \cap JP = IJ$.

(2) B is flat over A.

Proof. We need only observe that

$$\operatorname{Tor}_1^A(B, A/J) = (I \cap JP)/IJ.$$

Definition. If $E_1 \longrightarrow E$ and $E_2 \longrightarrow E$ are ring homomorphisms we define $E_1 \underset{E}{x} E_2$ as the set of ordered pairs $(x, y) \in E_1 \times E_2$ having the same projection in E. $E_1 \underset{E}{x} E_2$ is the fibred product in the category of rings, and $\operatorname{Spec}(E_1 \underset{E}{x} E_2)$ is the fibred direct sum of $\operatorname{Spec} E_1$ and $\operatorname{Spec} E_2$, in the category of affine schemes.

Proposition 2.

$$\begin{tikzcd}
B \arrow[r] \arrow[d] & B_1 \arrow[d] & & & \\
B_2 \arrow[r] & B_0 & & & \\
& & A \arrow[r] \arrow[d] \arrow[uull] & A_1 \arrow[d] \arrow[uull] & \\
& & A_2 \arrow[r] \arrow[uull] & A_0 \arrow[uull] & = A_1/J
\end{tikzcd}$$

Let A_1, A_2 be rings, J a nilpotent ideal in A_1, $A_0 = A_1/J$ and $A_2 \to A_0$ a morphism. Let B_i be a flat A_i algebra, $i=1,2$, and put $B_0 = B_2 \otimes_{A_2} A_0$. Let $B_1 \to B_0$ be an A_1 morphism inducing $B_1/JB_1 \xrightarrow{\sim} B_0$. Then $B = B_1 \times_{B_0} B_2$ is flat over $A = A_1 \times_{A_0} A_2$.

<u>Proof.</u> If we write $B_2 = P_2/I_2$ where $P_2 = A_2[X]$ is a polynomial ring over A_2, then $B_0 = P_0/I_0$, $B_1 = P_1/I_1$ with $P_0 = A_0[X]$ and $P_2 = A_2[X]$: Clearly $P = P_1 \times_{P_0} P_2 = A[X]$, $B/(J \times 0)B = B_2$, $P/I_1 \times_{P_0} I_2 = B$. Now $JI_1 = JP_1 \cap I_1$ by the corollary to Proposition 1, so that $(J \times 0)P \cap I_1 \times_P I_2 = (JI_1) \times 0 = (J \times 0)P \cdot (I_1 \times_P I_2)$, and B is flat over A, by the same corollary.

1.3. The Cotangent Complex and Affine Deformations.

1.3.1 Simplicity.

We recall the following definitions and facts from SGA II, III.

<u>Definition 1.</u> Let B be an A algebra. We say B is <u>simple</u> over A if

(i) A is noetherian, B is a localization of an A algebra of finite type and

(ii) B is flat over A, with fibres (absolutely) simple. circular !

<u>Definition 2.</u> Let $A \to B$ be a ring homomorphism.

We say B is <u>formally simple</u> over A if for every A algebra C and nilpotent ideal J in C, the map

$$\mathrm{Hom}_A(B, C) \longrightarrow \mathrm{Hom}_A(B, C/J)$$

is surjective.

Definition 3. Let B be a noetherian ring and I an ideal in B. We say B/I is a complete intersection in B if I may be generated by a B sequence. We say B/I is locally a complete intersection in B if $(B/I)_p$ is a complete intersection in B_p for any prime p in B.

If B and C are A algebras, with $B = C/I$ then we have an exact sequence

$$(*) \qquad I/I^2 \xrightarrow{d} \Omega_{C/A} \underset{C}{\otimes} B \longrightarrow \Omega_{B/A} \longrightarrow 0$$

Then we have the following:

Proposition 3. (SGA II)

Suppose in the above (*) that A is noetherian and C is a localization of an A algebra of finite type. Then

(i) (Jacobian Criterion) If C is simple over A then B is simple over A if and only if d is injective and $\Omega_{B/A}$ is a locally free B module.

(ii) If B is simple over A, then C is simple over A if and only if B is locally a complete intersection in C.

Proposition 4. [6] Let A be noetherian and B a localization of an A algebra of finite type. Then the following are equivalent

(i) B is simple over A.

(ii) $T^1(B/A, M) = (0)$ for all B modules M.

(iii) $T^1(B/A, k(p)) = (0)$ for all residue fields $k(p)$ of primes $p \in \operatorname{Spec} B$.

Proposition 5 [6]. Let $A \longrightarrow B$ be a ring homomorphism. Then B is formally simple over A if and only if $T^1(B/A, M) = (0)$ for all B modules M.

Proposition 6. [6]. Let C be a noetherian ring, and I an ideal in C, B = C/I. Then the following are equivalent:

(i) B is locally a complete intersection in C.

(ii) $T^2(B/C, M) = (0)$ for all B modules M.

(iii) $T^2(B/C, k(p)) = (0)$ for all primes p in Spec C.

As a corollary we have

Proposition 7 [6]. Let A be noetherian and B a localization of an A algebra of finite type. Then the following are equivalent

(i) For some representation B = C/I with C simple over A, B is locally a complete intersection in C.

(ii) For every representation as in (i), B is locally a complete intersection in C.

(iii) $T^2(B/A, M) = (0)$ for every B module M.

Under the equivalent conditions (i), (ii), (iii) of Proposition 7, we say that B is locally a complete intersection over A.

Proposition 8. Let A be noetherian, and B a localization of an A algebra of finite type. Write B = C/I with C simple over A. Then we have the exact sequence

$$\mathrm{Hom}_C(\Omega_{C/A}, M) \longrightarrow \mathrm{Hom}_C(I, M) \longrightarrow T^1(B/A, M) \longrightarrow 0$$

and the map $T^2(B/C, M) \longrightarrow T^2(B/A, M)$ is a bijection.

1.3.2. Algebra Extensions.

We note that if $A \longrightarrow B$ is a ring homomorphism and $B = P/I$ with P a polynomial ring over A then

$$T^0(B/A, M) = \mathrm{Hom}_B(\Omega_{B/A}, M) = \mathrm{Der}_A(B, M)$$

$$T^1(B/A, M) = \text{cokernel } \mathrm{Hom}_C(\Omega_{C/A}, M) \longrightarrow \mathrm{Hom}_C(I, M) \qquad C = P$$

(The second equation holds also if P is only formally simple over A.)

Definition. Let B be an A algebra, and M a B module.

Definition. By an extension of B/A by M we mean an exact sequence

$$(E) : 0 \longrightarrow M \xrightarrow{i} E \xrightarrow{j} B \longrightarrow 0$$

where E is a (commutative) A algebra, j is a homomorphism of A algebras, and M is regarded as a square zero ideal.

Let $(E) : 0 \longrightarrow M \xrightarrow{i'} E' \xrightarrow{j'} B \longrightarrow 0$ be another extension. We say that (E) and (E') are isomorphic if there exists a homomorphism $\theta : E \longrightarrow E'$ (of A algebras) which renders the diagram

$$\begin{tikzcd}
0 \arrow[r] & M \arrow[r, "i"] \arrow[d, equal] & E \arrow[r, "j"] \arrow[d, "\theta"] & B \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & M \arrow[r, "i'"'] & E' \arrow[r, "j'"'] & B \arrow[r] & 0
\end{tikzcd}$$

commutative. (θ must then be an isomorphism).

We define the M dual numbers over B to be the algebra $D_B(M)$ (or $D(M)$) $= M + B$ where $(m, b)(m', b') = (mb' + m'b, bb')$. The extension (E) is said to be trivial is it is isomorphic to $0 \longrightarrow M \xrightarrow{i} D(M) \xrightarrow{j} B \longrightarrow 0$

(im = (m, 0), j(b, 0) = b), i.e. if it has a section $B \overset{q}{\dashrightarrow} E_1$ such that $j \circ q = \mathrm{id}$.

We define Ex(B/A, M) to be the set of isomorphisms classes of extensions of B/A by M. In the diagram (E) above we let [E] denote the class of (E).

1.3.3. Variance of Ex(B/A, M).

(a) Change of rings.

Let
$$\begin{tikzcd} B' \arrow[r, "p"] & B \\ A' \arrow[u] \arrow[r] & A \arrow[u] \end{tikzcd}$$
be a commutative diagram of rings. Let M be a B' module. We define

$$p^* : \mathrm{Ex}(B/A, M) \longrightarrow \mathrm{Ex}(B'/A', M).$$

Given $(E) : 0 \longrightarrow M \xrightarrow{i} E \xrightarrow{j} B \longrightarrow 0$

then set $(p^*E) : 0 \longrightarrow M \xrightarrow[i']{} E' \xrightarrow[j']{} B' \longrightarrow 0$

where $E' = E \times_{B'} B$ and $i' = (i, 0)$. (p^*E) may be characterized up to isomorphism as the unique extension (E') of B'/A' for which there exists a "homomorphism of extensions" $E' \longrightarrow E$ restricting to id_M and P on the two ends.

(b) Change of modules. Let B be an A algebra, $\alpha : M \longrightarrow M'$ a homomorphism of B modules. We define

$$\alpha_* : \mathrm{Ex}(B/A, M) \longrightarrow \mathrm{Ex}(B/A, M')$$

as follows: Given

$$(E)^0\longrightarrow M \xrightarrow{i} E \xrightarrow{j} B \longrightarrow 0, \qquad \text{then set}$$

$$(\alpha_* E) : 0 \longrightarrow M' \xrightarrow[i']{} E' \xrightarrow[j']{} B \longrightarrow 0$$

where $E' = E \oplus M'/N$ and N is the module consisting of all pairs $(im, -\alpha m)$ for $m \in M$. Define $i'm'$ = class of $(0, m')$, $m' \in M'$. $(\alpha_* E)$ may be characterized up to isomorphism as the unique extension for which there exists a homomorphism $E \longrightarrow E'$ restricting to α and id_B on the two ends.

We now introduce, in the usual manner, a structure of B module on $Ex(B/A, M)$. Let $\nabla_B : B \oplus B \longrightarrow B$, and $\Delta_M : M \longrightarrow M \oplus M$ be defined by $\nabla_B(b, b') = b + b'$ and $\Delta_M(m) = (m, m)$. Given two extensions $[E]$ and $[E']$ in $Ex(B/A, M)$ define

$$[E + E'] = [\Delta_{M*}\nabla_B^*(E \oplus E')] = [\nabla_B^* \Delta_{M*}(E \oplus E')]$$

where $E \oplus E'$ is the obvious extension in $Ex_A(B \oplus B, M \oplus M)$. Then it can be checked that $Ex(B/A, M)$ becomes a commutative group, with the trivial extension $[D(M)] = 0$ and $-[E] = \alpha_*[E]$, where $\alpha = -id_M \in Hom(M, M)$.

It is also easy to check that α_* and p^* defined above are group homomorphisms, and that $M \rightsquigarrow Ex(B/A, M)$ is an additive functor from B modules to abelian groups. Thus $Ex(B/A, M)$ becomes a B module.

1.3.4. Extending homomorphisms.

Let $p : B \longrightarrow C$ be a homomorphism of A algebras, let M be a

C module, and let C' be an extension of C/A by M.

$$\begin{tikzcd}
B \arrow[rr, dashed, "p'"] \arrow[dr, "p"] & & C' \arrow[dl, "j"] \\
& C & \\
A \arrow[uu] \arrow[ur] & &
\end{tikzcd}$$

(1) There exists a homomorphism $p' : B \longrightarrow C'$ of A algebras such that $j \circ p' = p$ if and only if $p^*[C']$ $(= [C' \underset{C}{\times} B]) = 0$. (By the categorical definition of the fibred product.)

(2) If $C' = D(M)$ $(= C \oplus M)$ then $\theta \rightsquigarrow (p, \theta)$ is a bijection of $\mathrm{Der}_A(B, M)$ with the set of such p'.

In particular, if θ e $\mathrm{Der}_A(B, M)$ then we have an extension $\partial(\theta)$ in $\mathrm{Ex}(C/B, M)$, obtained by regarding $C \oplus M$ as a B algebra via $b \rightsquigarrow (pb, \theta b)$. Thus:

Proposition 9. Let $A \xrightarrow{p} B \xrightarrow{q} C$ be a sequence of ring homomorphisms, and M a C module. Then the sequence of C modules

$$\begin{aligned} (3)\quad & 0 \longrightarrow \mathrm{Der}_B(C, M) \longrightarrow \mathrm{Der}_A(C, M) \longrightarrow \mathrm{Der}_A(B, M) \\ & \xrightarrow{\partial} \mathrm{Ex}(C/B, M) \longrightarrow \mathrm{Ex}(C/A, M) \longrightarrow \mathrm{Ex}(B/A, M) \end{aligned}$$

is exact.

The proof is left to the reader.

1.3.5. Let B be an A algebra, and M a B module. If $[E]$ e $\mathrm{Ex}(B/A, M)$, then the sequence $A \longrightarrow E \longrightarrow B$ gives rise to a homomorphism

$$\mathrm{Hom}_B(M,\ M) = T^1(B/E,\ M) \xrightarrow{\rho} T^1(B/A,\ M)$$

Let id be the identity endomorphism of M then

Theorem 1.

(i) The assignment $(E) \rightsquigarrow \rho(\mathrm{id})$ induces an isomorphism of B modules

$$\alpha : \mathrm{Ex}(B/A,\ M) \xrightarrow{\sim} T^1(B/A,\ M)$$

(ii) If $A \to B \to C$ is a sequence of rings, and M a C module, then the diagram

$$\begin{tikzcd}
\mathrm{Der}_A(B,\ M) \arrow[r] \arrow[d, equal] & \mathrm{Ex}(C/B,\ M) \arrow[r] \arrow[d] & \mathrm{Ex}(C/A,\ M) \arrow[r] \arrow[d] & \mathrm{Ex}(B/A,\ M) \arrow[d] \\
T^0(B/A,\ M) \arrow[r] & T^1(C/B,\ M) \arrow[r] & T^1(C/A,\ M) \arrow[r] & T^1(B/A,\ M)
\end{tikzcd}$$

commutes.

Proof (i). Write $B = P/I$ where P is a polynomial ring over A. Recall that by definition of $T^1(B/A,\ M)$ we have an exact sequence

$$\mathrm{Der}_A(P,\ M) \xrightarrow{d^*} \mathrm{Hom}_B(I/I^2,\ M) \to T^1(B/A,\ M) \to 0$$

Now given $[E]$, we can lift $P \to B$ to $u : P \to E$ and induce a commutative diagram

$$\begin{tikzcd}
(\bar{P}) & 0 \arrow[r] & I/I^2 \arrow[r, "\bar{i}"] \arrow[d, "\bar{v}"] & P/I^2 \arrow[r, "\bar{j}"] \arrow[d, "\bar{u}"] & B \arrow[r] \arrow[d, equal] & 0 \\
(E) & 0 \arrow[r] & M \arrow[r, "i"'] & E \arrow[r, "j"'] & B \arrow[r] & 0
\end{tikzcd}$$

The class of $\bar{v}$ in $T^1(B/A,\ M)$ is independent of the choice of u; in fact it is equal to $\rho(\mathrm{id})$. If $\theta : E \to E'$ is an isomorphism of extensions then we may choose $u' : P \longrightarrow E'$ to be $\theta \circ u$, so that $\bar{v}' = \bar{v}$ and α is well defined. Given $\bar{v} : I/I^2 \longrightarrow M$ we can set β (class of $\bar{v}$) $= \bar{j}_*(\bar{v}) \in T^1(B/A, M)$ and get

$$\beta : T^1(B/A, M) \longrightarrow Ex(B/A, M)$$

In other words β sends $\bar{v}$ to the extension $[E]$, where E is $P/I^2 \oplus M$ factored out by the submodule of elements $(x, -\bar{v}(x))$, $x \in I/I^2$.

We leave it to the reader to check that α preserves the module structures and that β is the inverse to (α).

ii) Also left to the reader.

Remark. Theorem 1, together with observation (1) of §1.3.4 proves that B is formally simple over A if and only if $T^1(B/A, M) = (0)$ for all B modules M.

Convention. From now on we shall identify $\alpha[E]$ with $[E]$, and write $[E] \in T^1(B/A, M)$, etc.

1.3.6. Affine Deformations.

Definition. Let $A \longrightarrow B$ be a flat ring homomorphism, and let A' be a ring such that $A = A'/J$ with $J^2 = (0)$. By a deformation of B/A to A' we mean a flat A' algebra B' together with a morphism $B' \longrightarrow B$ inducing $B'/JB \xrightarrow{\sim} B$. In other words a deformation is a product diagram

$$\begin{tikzcd}
B' \arrow[r, dashed] & B \\
A' \arrow[u, dashed, "\text{Flat}"] \arrow[r] & A = A'/J \arrow[u, "\text{Flat}"']
\end{tikzcd}$$

Let Def $(B/A, A')$ be the set of isomorphism classes of deformations. Note that if B' is a deformation, then $J \otimes_{A'} B' \longrightarrow JB'$ is an isomorphism. Thus we find by the corollary to Proposition 1 that if $A' = D_A(J)$ $(= A \oplus J)$ then the map

$$[B'] \rightsquigarrow [0 \longrightarrow J \otimes_A B \longrightarrow B' \longrightarrow B \longrightarrow 0]$$

is a bijection of Def $(B/A, D(J))$ with $T^1(B/A, J \otimes_A \overset{B}{\cancel{M}})$.

Now consider the situation

$$\begin{tikzcd}
 & B \\
A' \arrow[r] & A = A/J \arrow[u, "\text{Flat}"']
\end{tikzcd}$$

where $J^2 = (0)$.

We have an exact sequence

$$0 \longrightarrow T^1(B/A, J \otimes_A B) \xrightarrow{u} T^1(B/A', J \otimes_A B) \xrightarrow{v} T^1(A/A', J \otimes_A B)$$
$$\xrightarrow{\partial} T^2(B/A, J \otimes_A B).$$

Note that $T^1(A/A', J \otimes_A B) = \mathrm{Hom}_B(J \otimes_A B, J \otimes_A B)$ has a distinguished element $\mathbf{1}$ corresponding to the identity endomorphism of $J \otimes_A B$ (or to the image of $[A']$ in

$$T^1(A/A', J) \longrightarrow T^1(A/A', J \underset{A}{\otimes} B) \Big),$$

Theorem 2. There exists a deformation of B/A to A' (i.e. $\mathrm{Def}(B/A, A') \neq \emptyset$) if and only if $\partial(\mathbb{1}) = 0$.

Proof. The question is whether there exists an extension

$$(E) \quad 0 \longrightarrow J \underset{A}{\otimes} B \xrightarrow{i} E \xrightarrow{j} B \longrightarrow 0$$

in $T^1(B/A', J \underset{A}{\otimes} B)$ which is actually a deformation of B/A to A'. To be precise, there exists an injection

$$\rho : \mathrm{Def}(B/A, A') \longrightarrow T^1(B/A', J \underset{A}{\otimes} B)$$

defined as follows. Given a deformation $B' \xrightarrow{p} B$, we have isomorphisms $J \underset{A'}{\otimes} B' \xrightarrow{s} J \underset{A}{\otimes} B$, and $J \underset{A'}{\otimes} B' \xrightarrow{t} JB'$. Set

$$\rho[B'] : 0 \longrightarrow J \underset{A}{\otimes} B \xrightarrow{tos^{-1}} B' \xrightarrow{p} B \longrightarrow 0 .$$

We claim that $[E]$ is in the image of ρ if and only if $v[E] = \mathbb{1}$.

In the first place, given an extension (E) as above we have a sequence of maps

$$J \underset{A'}{\otimes} E \xrightarrow{a} J \underset{A}{\otimes} B \xrightarrow{b} JE \xhookrightarrow{c} J \underset{A}{\otimes} B$$

where a is induced from j, and b is induced from $J \longrightarrow JE$; the inclusion c results from the fact that the image of JE in B is (0). The reader may check that $c \circ b = \nu[E]$.

Now if $[E] = \rho[B']$, then $a = s$, $b = t \circ s^{-1}$, $c = s \circ t^{-1}$, so that $c \circ b = \mathbb{1}$. Conversely, if $c \circ b = \mathbb{1}$, then c and $b \circ a$ are bijections, so that E is a deformation.

Remark. The obstruction $\partial(\mathbb{1})$ may be obtained more explicitly, but less canonically as follows. Write $B = P/I$ where P is a polynomial ring $A[X]$. Put $P' = A'[X]$. Let K be the inverse image of I in P'. The exact sequence $0 \longrightarrow JP' \longrightarrow K \longrightarrow I \longrightarrow 0$, and the map $\pi : JP' \longrightarrow J \otimes_A B$ induces the exact sequence

$$0 \longrightarrow J \otimes_A B \longrightarrow K/JK \longrightarrow I \longrightarrow 0$$

of P modules. Now the existence of an ideal $I' \subset K$ such that P'/I' is a deformation is equivalent to the existence of an $h : K \longrightarrow J \otimes_A B$ such that $h \,|\, JP' = \pi$. (Then we could put $I' = \{x + y \mid x \in K,\ \pi y = hx\}$.) In other words the obstruction is exactly the image of $\mathbb{1}$ under the connecting homomorphism

$$\mathrm{Hom}_P(J \otimes_A B,\ J \otimes_A B) \longrightarrow \mathrm{Ext}^1_P(I,\ J \otimes_A B)$$

Now in general, we have $T^2(B/A,\ M) \subset \mathrm{Ext}^1_P(I,\ M)$ for any B module M, and one checks that the image of $\mathbb{1}$ is actually in $T^2(B/A,\ J \otimes_A B)$.

§2. Deformations.

2.1. Preliminaries.

2.1.1. Extensions.

Definition 1. Let $X \longrightarrow S$ be a morphism of preschemes. By an *infinitesmal extension* of X/S we mean a closed immersion $X \hookrightarrow X'$ over S, defined by a nilpotent ideal in I in $\mathcal{O}_{X'}$. If $I^2 = (0)$, we say X' is a square zero extension of X/S. A morphism $X' \longrightarrow Y'$ of two infinitesmal extensions is just an S morphism which reduces to the identity on X. Thus we have the category of infinitesmal (resp. square zero) extensions of X/S.

Definition 2. Let $f : X \longrightarrow Y$ be an *affine* S morphism, and let $X \hookrightarrow X'$ be an infinitesmal (resp. square zero) extension of X/S. Then $Y \hookrightarrow Y' = Y \coprod_X X'$ is an infinitesmal (resp. square zero) extension of X/S, denoted by $f_*(X')$. (Here $\mathcal{O}_{Y'} = f_*\mathcal{O}_{X'} \times_{f_*\mathcal{O}_X} \mathcal{O}_Y$ is the sheaf given by $\mathcal{O}_{Y'}(U) = f_*\mathcal{O}_{X'}(U) \times_{f_*\mathcal{O}_x(U)} \mathcal{O}_Y(U)$ for U open in Y.)

Note that a square zero extension of X/S is just an exact sequence

$$(\mathcal{E}) \; : \; 0 \longrightarrow I \xrightarrow{i} \mathcal{E} \xrightarrow{j} \mathcal{O}_X \longrightarrow 0$$

where I is an $\mathcal{O}_X$ module and j is a homomorphism of sheaves of $\mathcal{O}_S$ modules. We define two such extensions $(\mathcal{E})$ and $(\mathcal{E}')$ to be isomorphic if there is a homomorphism $\alpha : \mathcal{E} \to \mathcal{E}'$ which reduces to the identity on both $\mathcal{O}_X$ and I. (α must then be an isomorphism.) Given an $\mathcal{O}_X$ module I we define the *dual I numbers over X* to be the prescheme $(X, \mathcal{O}_X \oplus I)$ denoted by $D_X(I)$, or just $D(I)$ if no confusion can result.

Thus we have in particular the extension

$$(D_X(I)) : 0 \to I \to \mathcal{O}_X \oplus I \to \mathcal{O}_X \to 0.$$

The extension $(\mathcal{E})$ is isomorphic to $(D_X(I))$ if and only if there is a section $j' : \mathcal{O}_X \to \mathcal{E}$ such that $j \circ j' =$ identity.

Definition 3. Let $X \to S$ be a morphism, and I a sheaf of $\mathcal{O}_X$ modules. We define $\mathrm{Ex}(X/S, I)$ to be the set of isomorphism classes of square zero extensions of X/S with kernel I, using the above notion of isomorphism.

Remark. If X' and Y' are two square zero extensions of X/S, both having the same $\mathcal{O}_X$ module I as kernel, then an isomorphism $\alpha : X' \to Y'$ in the category of square zero extensions of X/S is not an isomorphism of square zero extensions with kernel I, unless α leaves I fixed.

Thus we find, exactly as in the affine case, that $\mathrm{Ex}(X/S, I)$ is an abelian group, and that $I \rightsquigarrow \mathrm{Ex}(X/S, I)$ is an additive functor from $\mathcal{O}_X$ modules to abelian groups. (Esp. $\mathrm{Ex}(X/S, I)$ is a $\Gamma(X, \mathcal{O}_X)$ module.) Further:

Proposition 1. Let

$$\begin{tikzcd} X \arrow[r, "p"] \arrow[dr] & Y \arrow[d, "q"] \\ & Z \end{tikzcd}$$

be a commutative diagram of preschemes with p affine. Let I be an $\mathcal{O}_X$ module. Then there is an exact sequence

$$\begin{aligned} &0 \to \mathrm{Der}_{\mathcal{O}_Y}(\mathcal{O}_X, I) \to \mathrm{Der}_{\mathcal{O}_Z}(\mathcal{O}_X, I) \to \mathrm{Der}_{\mathcal{O}_Z}(\mathcal{O}_Y, p_*I) \\ (1) \qquad &\xrightarrow{\partial} \mathrm{Ex}(X/Y, I) \xrightarrow{q_*} \mathrm{Ex}(X/Z, I) \xrightarrow{p_*} \mathrm{Ex}(Y/Z, p_*I) \end{aligned}$$

(Der stands for the global sections of the sheaf of derivations, ∂ is the obvious globalization of the coboundary homomorphism defined in §1.3.4; the rest of the maps should be self explanatory).

Proof. Straightforward checking. Finally we shall need the following on flat change of base:

Proposition 2. Let $X \longrightarrow S$ be a flat morphism, $T \longrightarrow S$ an affine morphism, and $Y = X \underset{S}{\times} T$. Let J be an $\mathcal{O}_Y$ module.

$$\begin{tikzcd}
X \underset{S}{\times} T = Y \arrow[r, "p"] \arrow[d] & X \arrow[d, "\text{Flat}"] \\
T \arrow[r] & S
\end{tikzcd}$$

Then

$$p_* : \mathrm{Ex}(Y/T,\ J) \longrightarrow \mathrm{Ex}(X/S,\ p_*J)$$

is a bijection. ($p_*(\mathcal{F}) = \mathcal{F} \underset{Y}{\times} X$, for $\mathcal{F}$ e $\mathrm{Ex}(Y/T,\ J)$).

Proof. Let $\alpha : p^* p_* J \longrightarrow J$. If $(\mathcal{E})$ e $\mathrm{Ex}(X/S,\ p_*\mathbb{J})$, then $(\mathcal{E} \underset{\mathcal{O}_X}{\times} \mathcal{O}_Y)$ e $\mathrm{Ex}(Y/T,\ p^*p_*I)$ by flatness. Define

$$q^* : \mathrm{Ex}(X/S,\ p_*J) \longrightarrow \mathrm{Ex}(Y/T,\ J)$$

By
$$q^*[\mathcal{E}] = \alpha_*[\mathcal{E} \underset{\mathcal{O}_X}{\otimes} \mathcal{O}_Y] .$$

We claim q^* is the inverse to p_*. In fact, we have, for $[\mathcal{E}]$ e $\mathrm{Ex}(X/S,\ p_*J$, $[\mathcal{F}]$e $\mathrm{Ex}(Y/T,\ J)$ homomorphisms

$$\mathcal{E} \longrightarrow p_*q^*\mathcal{E}$$

and

$$q^* p_* \mathcal{F} \longrightarrow \mathcal{F}$$

which are easily seen to be isomorphisms.

Note that by EGA I, 5.1.9, a prescheme X is affine if and only if X_{red} is affine. Thus if $S = \operatorname{Spec} A$, $X = \operatorname{Spec} B$, and $I = \tilde{M}$ where M is a B module, then

$$\operatorname{Ex}(B/A,\ M) = \operatorname{Ex}(X/S,\ I).$$

§2.2.1. Formally principal homogeneous spaces.

Let $\underline{C}$ be a category in which products exist; let P be an object of $\underline{C}$ and G a group object. We say that G operates on the right of P if there is a morphism $\pi : P \times G \longrightarrow P$ having the obvious properties, (i.e. inducing, for each object T a group action of $G(T)$ on $P(T)$ in a way compatible with morphism $T^1 \longrightarrow T$.) We say P is a formally principal homogeneous space if the morphism $(\pi,\ pr_1) : P \times G \longrightarrow P \times P$ is an isomorphism.

Now let $X \longrightarrow S$ be a morphism, and let $\underline{C}$ be the category of square zero extensions of X/S. If $Y \in$. $(\underline{C})$ has kernel I we define a morphism

$$\pi_Y : Y \longrightarrow Y \coprod_X D(I)$$

in $\underline{C}$ (or: $(\mathcal{O}_X \oplus I) \times_{\mathcal{O}_X} \mathcal{O}_Y \longrightarrow \mathcal{O}_Y$). By

$$(f + g,\ h) \rightsquigarrow g + h$$

For $f \in \mathcal{O}_X(U)$, $g \in I(U)$, $h \in \mathcal{O}_Y(U)$, U open in X.

Proposition 2.

(i) $\pi_{D(I)}$ makes $D(I)$ a commutative group object in the opposed category $\underline{C}^{o}$.

(ii) If $Y \in \ (\underline{C})$ has kernel I, then π_Y makes Y a formally principal homogeneous object under $D(I)$, in $\underline{C}^{o}$.

Proof. $\underline{C}^{o}$ is just the category of algebra extensions

$$(\mathcal{E}) \qquad 0 \longrightarrow I \rightarrow \mathcal{E} \rightarrow \mathcal{O}_X \longrightarrow 0$$

so we work with these. The inverse mapping $\mathcal{O}_D \longrightarrow \mathcal{O}_D$ is given by $(f+g) \rightsquigarrow (f-g)$, notation as above. If $\mathcal{E} \in \underline{C}$ has kernel I, the diagram

$$\begin{tikzcd}
 & \mathcal{O}_D \times_{\mathcal{O}_X} \mathcal{E} \arrow[dr, "\pi_{\mathcal{E}}"] & \\
\mathcal{O}_D \times_{\mathcal{O}_X} \mathcal{O}_D \times_{\mathcal{O}_X} \mathcal{E} \arrow[ur, "{(\pi_D, 1)}"] \arrow[dr, "{(1, \pi_{\mathcal{E}}}"'] & & \mathcal{E} \\
 & \mathcal{O}_D \times_{\mathcal{O}_X} \mathcal{E} \arrow[ur, "\pi_{\mathcal{E}}"'] &
\end{tikzcd}$$

is commutative, since $(g_1 + g_2) + h = g_1 + (g_2 + h)$ $\Big(g_i \in I(U),\ h \in \ (U),\ U \text{ open}\Big)$ Thus we see that $\mathcal{O}_D$ is a group object, and that $\mathcal{O}_D \times \mathcal{E} \longrightarrow \mathcal{E}$ is a group operation. The fact that $\mathcal{O}_D \times_{\mathcal{O}_X} \mathcal{E} \longrightarrow \mathcal{E} \times_{\mathcal{O}_X} \mathcal{E}$ is an isomorphism may be easily checked by taking sections over an open.. U.

Corollary. Let $\underline{C}'$ be a category in which products exist. Let $F : \underline{C}^{o} \longrightarrow \underline{C}'$ be a functor which commutes with products, (i.e., F is a contravariant functor from square zero extensions to $\underline{C}'$ such that $F(Y \amalg_X Z) \longrightarrow F(Y) \times F(Z)$

is an isomorphism). Then for any sheaf I on X,

(1) F(D(I)) is a commutative group object.

(2) If $Y \in$ $(\underline{C})$ has kernel I, then F(Y) is a formally principal homogeneous object under F(D(I)).

2.2. Formal Theory of Deformations.

2.2.0. Throughout this section 2.2, we shall assume for convenience that all preschemes are locally noetherian <u>schemes</u>, and that all morphisms are locally of finite type. According to the results quoted in §1.1, the functors T^i globalize; given a morphism. $X \longrightarrow S$ and a coherent sheaf F on X, $T^i(X/S, F)$ is a coherent sheaf on X, for i = 0, 1, 2. In order to globalize the exact sequence (6) of §1.1 we shall need to consider the following situation:

(1)
$$\begin{tikzcd} F, X \arrow[r, "f"] \arrow[dr] & Y \arrow[d] \\ & S \end{tikzcd}$$

Let $f : X \longrightarrow Y$ be a morphism over S, and F a coherent $\mathcal{O}_X$ module. Let (U_i) be an affine open cover of Y, where $U_i = \operatorname{Spec} B_i$ lies over $\operatorname{Spec} A_i \subset S$, and let $V_i = f^{-1}(U_i)$. Let L_i be a cotangent complex for B_i/A_i and set

$$T^j(Y/S, F) | V_i = H^j(\underline{\mathrm{Hom}}_{\mathcal{O}_X}(f^*\tilde{L}_i, F|V_i))$$

j = 0, 1, 2. According to P.2 (1),(3) of §1.1, these glue together to form a coherent sheaf $T^j(Y/S, F)$ on X, and thus we have an exact sequence

$$\begin{aligned} & 0 \to T^0(X/Y, F) \to T^0(X/S, F) \to T^0(Y/S, F) \\ (2) \qquad & \to T^1(X/Y, F) \to T^1(X/S, F) \to T^1(Y/S, F) \\ & \to T^2(X/Y, F) \to T^2(X/S, F) \to T^2(Y/S, F) \end{aligned}$$

or coherent sheaves on X, for any diagram such as (1).

Now let $A \to B$ be a ring homomorphism, and M a B module. In view of the commutative diagram on p.12 of §1.3.5, we know that the identification $T^1(B/A, M) = \mathrm{Ex}(B/A, M)$ commutes with localization.

Thus if we are given a morphism $X \to S$ and a coherent $\mathcal{O}_X$ module F, we find that

(3) $T^1(X/S, F)$ = sheaf of germs of extensions of X/S by F, where the right hand side of (3) is the sheaf assoiated to the presheaf

$$U \rightsquigarrow \mathrm{Ex}(U/S, F/U), \; U \text{ open in } X.$$

In particular, given an extension $[\mathcal{E}] \in \mathrm{Ex}(X/S, F)$ we may associate with it a section $\ell[E]$ of $T^1(X/S, F)$ whose restriction to affine $U \subset X$ is the class of $\mathcal{E}|U$. Thus we have an "edge homomorphism".

$$(4) \qquad \ell : \mathrm{Ex}(X/S, F) \longrightarrow H^0(X, T^1(X/S, F))$$

called the local class map.

2.2.2. Infinitesmal extension of morphisms.

Consider a commutative diagram

(5)
$$\begin{tikzcd} X \arrow[d] & Y \arrow[l, "f"'] \arrow[r, hook, "i"] \arrow[dl] & Z \arrow[dll] \\ S & & \end{tikzcd}$$

where $i : Y \hookrightarrow Z$ is an infinitesmal extension of Y/S. We denote by $\underline{E}^Z$ the sheaf of sets on Y whose sections over an open $U \subseteq Y$ are the extensions of $f|U$ to $Z|U$:

(6)
$$\underline{E}^Z(U) = \{g : Z|U \to X \mid g \circ i|U = f|U\}$$

On the other hand, given any coherent sheaf J on Y we have the exact sequence

(7)
$$T^0(X/S, J) \to T^1(Y/X, J) \to T^1(Y/S, J) \xrightarrow{\alpha} T^1(X/S, J)$$

<u>Proposition 3.</u> Suppose in diagram (5), that Z is a square zero extension of Y/S with kernel J. Then

(i) $\underline{E}^Z$ is a formally principal homogeneous sheaf under $T^0(X/S, J)$ $(= \underline{\mathrm{Hom}}(f^*\Omega_{X/S}, J))$.

(ii) Let $c^o = \alpha_* \ell[Z] \in H^o(Y, T^1(X/S, J))$, and let $y \in Y$. Then $(\underline{E}^Z)_y \neq \emptyset$ if and only if $c^o_y = 0$.

(iii) If $c^o = 0$, then there is an element $c^1 \in H^1(X, T^o(X/S, J))$ with the property that $\Gamma(Y, \underline{E}^Z) \neq \emptyset$ if and only if $c^1 = 0$.

In (ii), (resp. (iii)) the condition $\underline{E}^Z_y \neq \emptyset$ (resp. $\Gamma(Y, \underline{E}^Z), \neq \emptyset$) means that f can be extended to $Z|U$ for sufficiently small $U \ni y$ (resp. f can be extended to Z).)

<u>Proof.</u> (i) Let $\underline{C}$ be the category of square zero extensions of Y/S and

$\underline{C}'$ the category of sheaves of sets on Y. Then the functor $Z \rightsquigarrow \underline{E}^Z$ from $\underline{C}^{\circ} \longrightarrow \underline{C}'$ clearly satisfies the hypotheses of the corollary to Proposition 2, so we have only to identify $\underline{E}^D$ where $D = D(J) = \mathcal{O}_Y \oplus J$. It suffices to show that $\underline{E}^D(Y) = H^{\circ}(Y, \underline{\mathrm{Hom}}_{\mathcal{O}_X}(f^*\Omega_{X/S}, J))$. Now an extension of f to D is just a section h of $Pr_1 : D \underset{S}{\times} X \longrightarrow D$ such that

$$\begin{tikzcd}
h : D \arrow[r, dashed] \arrow[d, dashed, shift right, "j"'] & D \underset{S}{\times} X \\
Y \arrow[u, hook, shift right, "i"'] \arrow[ur, "{(f, i)}"'] &
\end{tikzcd}$$

$$\begin{tikzcd}
\mathcal{O}_{D\times_S X} \arrow[r, "I \quad h_0^\#"] \arrow[dr] & \mathcal{O}_D \arrow[dl, "\text{canonical}"'] \\
\mathcal{O}_Y & 0
\end{tikzcd}$$

commutes. Let h_0 be the section of $D \underset{S}{\times} X \longrightarrow D$ deduced from the canonical section $j : D \dashrightarrow Y$ of i, and let I be the ideal in $\mathcal{O}_{D \underset{S}{\times} X}$ defining h_0. Then since $J^2 = (0)$, any desired section h must factor through $D' = (D, \mathcal{O}_D + I/I^2)$:

$\mathcal{O}_D + I = \mathcal{O}_D \otimes_{\mathcal{O}_S} \mathcal{O}_X$

$$\begin{tikzcd}
D \arrow[r, shift left, "h'_0"] \arrow[r, dashed, shift right, "h'"'] & D' \\
Y \arrow[u, hook, "i"] \arrow[ur, hook, "{(f, i)'}"'] &
\end{tikzcd}$$

$g_0 i = f$

Thus the desired sections h' are sections of $\mathcal{O}_D \longrightarrow \mathcal{O}_{D'}$, and may be identified with $\mathrm{Hom}_{\mathcal{O}_D}(I/I^2, J) = \mathrm{Hom}_{\mathcal{O}_Y}(i^*I/I^2, J)$. On the other hand $h_0 : D \longrightarrow D \underset{S}{\times} X$ is deduced from the diagonal section $X \longrightarrow X \underset{S}{\times} X$ of $Pr_1 : X \underset{S}{\times} X \longrightarrow X$ by the base extension $g_0 : D \longrightarrow X$ ($g_0 = f \circ j$) so that $g_0^*(\Omega_{X/S}) = I/I^2$, and $i^*I/I^2 = f^*\Omega_{X/S}$. Q.E.D.

(ii) We know that for S = Spec A, X = Spec B, Y = Spec C and Z = Spec C':

$$\begin{tikzcd}
B \arrow[dr] \arrow[drr] & & \\
 & C \arrow[r, shift right] & C' \arrow[l, shift right, dashed] \\
A \arrow[uu] \arrow[ur] \arrow[urr] & &
\end{tikzcd}$$

The obstruction to lifting $B \to C$ to $B \to C'$ is $[B \underset{C}{\times} C'] \in T^1(B/A, J)$. But $[B \underset{C}{\times} C'] = \alpha[C']$. Hence the result.

(iii) This follows from the general theory of formally principal homogeneous spaces. Let $\underline{G} = T^{0}(X/S, J)$. For a suitable open cover (U_i) we have $f_i \in \underline{E}^2(U_i)$. Then there exist $\sigma_{ij} \in \underline{G}(U_i \cap U_j)$, such that $(f_i | U_{ij})^{\sigma_{ij}} = f_j | U_{ij}$. (σ_{ij}) is the cocycle c^1. ((iii) may also be reasoned out from the exact sequence (7)).

<u>Remark.</u> It seems that the obstructions c^0 and c^1 should be subsumed in a single obstruction lying in $Ex(X/S, f_* J)$. However we cannot see how to find one there unless f is affine.

2. 2. 3. Deformations.

<u>Definition.</u> Let $X \to S$ be a <u>flat</u> morphism. Let $S \hookrightarrow T$ be an infinitesimal extension of S (over Spec $\mathbb{Z}$). By a deformation of X/S to T we mean a <u>flat</u> scheme Y over T, together with a closed immersion $X \hookrightarrow Y$ over T inducing $X \xrightarrow{\sim} Y \underset{T}{\times} S$. In other words, a deformation of X/S to T is a product diagram

$$\begin{tikzcd}
X \arrow[r, hook] \arrow[d] & Y \arrow[d] & \text{flat} \\
S \arrow[r, hook] & T &
\end{tikzcd}$$

Two deformations are <u>isomorphic</u>* if they are isomorphic as product diagrams. Define

---

*Any T-morphism $Y_1 \to Y_2$ of deformations which reduce to the identity on X is an isomorphism.

$\mathrm{Def}(X/S,\ T)$ = Set of isomorphism classes of deformations.

$\underline{\mathrm{Def}}(X/S,\ T)$ = Sheaf of germs of deformations

= Sheaf associated to the presheaf $U \rightsquigarrow \mathrm{Def}(U/S,\ T)$, U open in X.

If Y is a deformation, we denote its class in $\mathrm{Def}(X/S,\ T)$ by $[Y]$. We have the usual map $\mathcal{L}\ : \mathrm{Def}(X/S,\ T) \longrightarrow H^0(X,\ \underline{\mathrm{Def}}(X/S,\ T))$ which assigns to each deformation its local class.

If Y is a deformation of X/S to T we denote by $\mathrm{Aut}_T(Y/X)$ the group of automorphisms Y reducing to the identity on X, and by $\underline{\mathrm{Aut}}_T(Y/X)$ the sheaf $U \rightsquigarrow \underline{\mathrm{Aut}}_T(Y|U/X)$. As a corollary to Proposition 3, we have:

<u>Proposition 4.</u> Let T be a square zero extension of S with kernel J, and let Y be a deformation of X/S to T. Then there are canonical isomorphisms

$$\mathrm{Aut}_T(Y/X) \approx H^0(X,\ T^0(X/S,\ J \otimes \mathcal{O}_X))$$

$$\underline{\mathrm{Aut}}_T(Y/X) \approx T^0(X/S,\ J \otimes \mathcal{O}_X)$$

Now let X be flat over S, let T_1 and T_2 be infinitesmal extensions of S and let Y_i be a deformation of X/S to $T_i (i = 1,\ 2)$. According to Proposition 2 of §1. 2, $Y_1 \amalg_X Y_2$ is a deformation of X/S to $T_1 \amalg_S T_2$. On the other hand, if Y is any deformation of X/S to $T_1 \amalg_S T_2$ which pulls back to Y_i by $T_i \longrightarrow T_1 \amalg_S T_2$, then we have a morphism $Y_1 \amalg_X Y_2 \longrightarrow Y$ over $T_1 \amalg_S T_2$ which must then be an isomorphism. A similar argument holds for local deformations. Thus the functors $T \rightsquigarrow \mathrm{Def}(X/S, T)$ and $T \rightsquigarrow \underline{\mathrm{Def}}(X/S, T)$ commute with fibred direct sums. We conclude by the corollary to Proposition 2 that if T is an extension of S with kernel J that $\mathrm{Def}(X/S, T)$ (resp. $\underline{\mathrm{Def}}(X/S, T)$) is a formally principal homogeneous space (resp. sheaf) under $\mathrm{Def}(X/S, D(J))$ (resp. $\underline{\mathrm{Def}}(X/S, D(J))$.

On the other hand the map

$$[Y] \rightsquigarrow [0 \longrightarrow J \otimes \mathcal{O}_X \longrightarrow \mathcal{O}_Y \rightarrow \mathcal{O}_X \longrightarrow 0]$$

and a similar map for the local deformations produce maps

$$(8) \qquad \begin{gathered} \mathrm{Def}(X/S, D(J)) \longrightarrow \mathrm{Ex}(X/S, \mathcal{O}_X \otimes J) \\ \underline{\mathrm{Def}}(X/S, D(J)) \longrightarrow T^1(X/S, \mathcal{O}_X \otimes J) \end{gathered}$$

which may be shown to be isomorphisms, commuting with the respective "local class" maps ℓ.

Now let $X \longrightarrow S$ be a morphism, and I a coherent sheaf on X. We claim that there is an exact sequence

$$(9) \quad 0 \longrightarrow H^1(X, T^0(X/S, I)) \xrightarrow{\tau} \mathrm{Ex}(X/S, I) \xrightarrow{\ell} H^0(X, T^1(X/S, I)) \xrightarrow{\rho} H^2(X, T^0(X/S, I)$$

The sequence (9) actually follows formally from the fact that if $(\mathcal{E})$ is an extension of X/S by I, then $\underline{G} = T^0(X/S, I)$ is canonically identified as the sheaf of automorphisms of $\mathcal{E}$ (leaving I and $\mathcal{O}_X$ fixed). We outline briefly the details.

To define τ, let (U_i) be a covering of X by affine opens, let $\sigma = (\sigma_{ij})$ be a Čech one cocycle for the cover (U_i). Let $\mathcal{E}$ be the sheaf of algebras obtained by patching together the sheaves $\mathcal{E}_i = D(I)|U_i$ via the transition isomorphisms

$$\sigma_{ij} : D(I)\,|U_i \cap U_j \xrightarrow{\sim} D(I)\,|\,U_i \cap U_j$$

Set $\tau(\text{class }(\sigma)) = [\mathcal{E}]$. Then τ is easily seen to be an isomorphism of $H^1(X, \underline{G})$ onto the subgroup of "locally trivial" extensions of X/S by I. ℓ is the "local class" map, as above. To define ρ, let $h \in H^0(X/S, T^1(X/S, I))$. Then for a suitable affine open cover (U_i) of X, h corresponds to a collection of extensions $\mathcal{E}_i$ of $X|U_i$ over S by $I|U_i$. Since h is a global section, there exist isomorphisms $\sigma_{ij} : \mathcal{E}_i|U_i \cap U_j \xrightarrow{\sim} \mathcal{E}_j|\,U_i \cap U_j$. Now the $\mathcal{E}_i$ patch together (i.e., h is in the image of ℓ) if and only if we can find new isomorphisms σ'_{ij} which satisfy the transitivity condition $\sigma'_{ij} \circ \sigma'_{jk} = \sigma'_{ik}$ on $U_i \cap U_j \cap U_k$. Such a σ'_{ij} will have the form

$$(10) \qquad \sigma'_{ij} = \sigma_{ij} \circ \theta_{ij}$$

where $\theta_{ij} \in \underline{G}(U_i \cap U_j)$ is an automorphism of $\mathcal{E}_i|U_i \cap U_j$, and conversely any one cochain $\theta = (\theta_{ij})$ defines a σ'_{ij} via (10). The transivity condition (10) now reads

$$(11) \qquad \theta_{ij} \circ \theta_{jk} \circ \theta_{ik}^{-1} = \sigma_{ik} \circ \sigma_{jk}^{-1} \circ \sigma_{ij}^{-1} \qquad \text{on } U_{ijk}$$

But the right hand side of (11) is an automorphism of $\mathcal{E}_i|U_{ijk}$, i.e. an element of $\underline{G}(U_{ijk})$ which is seen to form a two cocycle $\sigma = (\sigma_{ijk})$ of $\underline{G}$ whose class in $H^2(X, T^0(X/S, I))$ is independent of the choice of the isomorphisms σ_{ij}. We define ρ(~~class of~~ h) = class of σ. Recalling that addition in $\underline{G}$ corresponds to composition in Aut, we see that equation (11) may be written

$$\delta\theta = \sigma \tag{12}$$

Thus the exactness of (9) follows. (We leave to the reader the chore of proving that the maps τ, ρ are independent of the coverings (U_i) chosen.)

We now consider the problem of existence of deformations. Suppose we are given a diagram

$$\begin{tikzcd} X \arrow[d, "\text{flat}"'] & \\ S \arrow[r, hook] & T \end{tikzcd}$$

where T is a square zero extension of S with kernel J. Put $I = \mathcal{O}_X \underset{S}{\otimes} J$ and $\underline{G} = T^o(X/S, I)$. We have an exact sequence

$$0 \to T^1(X/S, I) \to T^1(X/T, I) \to T^1(\underset{S/T}{\cancel{T/S}}, I) \xrightarrow{\partial} T^2(X/S, I) \tag{13}$$

where $T^1(T/S, I) = \underline{\mathrm{Hom}}_{\mathcal{O}_X}(I, I)$.

Let $\mathbb{1} \in H^o(X, T^1(X/S, I))$ be the corresponding identity section. By Theorem 2 of §1.3, (p. 15) the element $C^{0,2} = \partial_* \mathbb{1}$ in $H^o(X, T^2(X/S, I))$ is exactly the obstruction to the existence of a local deformation. (In other words $C_x^{0,2} = 0$ if and only if $\underline{\mathrm{Def}}(X/S, T)_x \neq \emptyset$, where $x \in X$.) If $C^{0,2} = 0$, then $\underline{\mathrm{Def}}(X/S, T)$ is a formally principal homogeneous space under $T^1(X/S, I)$ which is locally principal (i.e. non empty), and therefore we get an element $C^{1,1}$ in $H^1(X, T^1(X/S, T))$ which vanishes if and only if $H^0(X, \underline{\mathrm{Def}}(X/S, T)) \neq \emptyset$. (The latter condition means that there is a global germ of a deformation, i.e., a cover (U_i), deformation Y_i of $X|U_i$ and isomorphisms $Y_i|U_{ij} \xrightarrow{\sim} Y_j|U_{ij}$.)

Finally if $C^{0,2}$ and $C^{1,1}$ are 0, we have an "exact" sequence

$$(14) \quad \mathrm{Def}(X/S, T) \xrightarrow{\ell_T} H^0(X, \underline{\mathrm{Def}}(X/S, T)) \xrightarrow{\rho_T} H^2(X, T^0(X/S, I))$$

which "commutes" with the sequence (9). For $h \in H^0(X, \underline{\mathrm{Def}}(X/S, T))$ we have that $\ell_T^{-1}(h)$ is a formally principal homogeneous space under $H^1(X, T^0(X/S, I))$, and $C^{2,0}(h) = \rho_T(h) = 0$ if and only if $\ell^{-1}(h) \neq \emptyset$. If $C^{0,2} = C^{1,1} = 0$, and $C^{2,0}(h) = 0$ for some h, then $\mathrm{Def}(X/S, T)$ is a principal homogeneous space under $\mathrm{Ex}(X/S, I) = \mathrm{Def}(X/S, D(J))$. Note that the image of ρ_T must be a coset of the image of ρ in (9).

We collect these details in:

Theorem 1. Given a diagram

$$\begin{tikzcd} X \arrow[d, "\text{flat}"'] & \\ S \arrow[r, hook] & T \end{tikzcd}$$

where T is a square zero extension of S with kernel J. Put $I = J \otimes_{\mathcal{O}_S} \mathcal{O}_X$. Then

(i) $\mathrm{Def}(X/S, T)$ is a formally principal homogeneous space under $\mathrm{Ex}(X/S, I)$.

$\underline{\mathrm{Def}}(X/S, T)$ is a formally principal homogeneous space under $T^1(X/S, I)$.

(ii) We have a "commutative" diagram

$$\begin{tikzcd}
0 \arrow[r] & H^1(X, T^o(X/S,I)) \arrow[r, "\tau"] & \mathrm{Ex}(X/S,I) \arrow[r, "\ell"] & H^o(X, T^1(X/S,I) \arrow[r, "\rho"] & H^2(X, T^o(X/S,I)) \\
 & & \mathrm{Def}(X/S, T) \arrow[r, "\ell_T"] & H^o(X, \underline{\mathrm{Def}}(X/S, T)) \arrow[ur, "\rho_T"'] &
\end{tikzcd}$$

(meaning that ℓ_T and ρ_T commute with the appropriate group operations) where the top row is exact. For $h \in H^o(X, \underline{\mathrm{Def}}(X/S, T))$, $\ell_T^{-1}(h)$ is a formally principal homogeneous space under $H^1(X, T^o(X/S, I))$.

(iii) If we form the exact sequence

$$0 \longrightarrow T^1(X/S, I) \longrightarrow T^1(X/T, I) \longrightarrow T^1(T/S, I) \xrightarrow{\partial} T^2(X/S, I)$$

and let $\mathbb{1}$ be the distinguished element of $T^1(T/S, I) = \underline{\mathrm{Hom}}_{\mathcal{O}_X}(I, I)$ corresponding to the identity homomorphism, then $C^{0,2} = \partial_*(\mathbb{1}) \in H^o(X, T^2(X/S, I))$ is zero if and only if $\underline{\mathrm{Def}}(X/S, T)$ is locally non empty.

(iv) If $C^{0,2} = 0$ then there exists a $C^{1,1} \in H^1(X, T^1(X/S, T))$ such that $C^{1,1} = 0$ if and only if $H^o(X, \underline{\mathrm{Def}}(X/S, T)) \neq \emptyset$.

(v) If $C^{0,2} = C^{1,1} = 0$, then $\mathrm{Def}(X/S, T) \neq \emptyset$ if and only if $\not{C}^{2,0}(h) = \rho_T(h)$ is zero for some $h \in H^o(X, \underline{\mathrm{Def}}(X/S, T))$.

<u>Remark. 1.</u> It seems here again that the obstructions $C^{0,2}$, $C^{2,0}$ and $C^{1,1}$ ought to be subsumed in single obstruction C^2 lying in a global $\mathrm{Ex}^2(X/S, I)$, and that, even more, there should be higher Ex^n's and T^n's and a spectral sequence

$$H^p(X, T^q(X/S, I)) \Longrightarrow \mathrm{Ex}^n(X/S, I)$$

generalizing (9). In fact, if $A \longrightarrow B$ is a ring homomorphism, then $T^2(B/A, M)$ may be interpreted as a certain "two term extensions" of B/A, by M, and this interpretation gives a candidate for $\mathrm{Ex}^2(X/S, I)$). We hope to clear up these matters in the future.

Remark 2. The sequence in (ii) above is our only control on the size of $\mathrm{Ex}(X/S, I)$. $T^1(X/S, I)$ may be conveniently computed by imbedding X in a simple scheme over S.

Remark 3. In general we have exact sequences

$$\text{(a)}\quad 0 \to T^1(X/S, I) \to \underline{\mathrm{Ext}}^1_{\mathcal{O}_X}(\Omega_{X/S}, I) \to \underline{\mathrm{Hom}}_{\mathcal{O}_X}(T_1(X/S, \mathcal{O}_X), I)$$

$$\text{(b)}\quad 0 \to T^2(X/S, I) \to \underline{\mathrm{Ext}}^1_{\mathcal{O}_X}(K/K^2, I) \to \underline{\mathrm{Hom}}_{\mathcal{O}_X}(T_2(X/S, \mathcal{O}_X), I)$$

where in (b), K/K^2 is the conormal sheaf defined by an immersion $X \hookrightarrow Y$ into a simple Y/S, with $\mathcal{O}_X = \mathcal{O}_Y/K$. Thus if S is integral, X is reduced, $X \to S$ is generally simple and dominant, then $T^1(X/S, I) = \underline{\mathrm{Ext}}^1_{\mathcal{O}_X}(\Omega_{X/S}, I)$ and $T^2(X/S, I) = \underline{\mathrm{Ext}}^2_{\mathcal{O}_Y}(\mathcal{O}_X, I)$ (see [6]). The sequence in (ii) then becomes the first four terms of the spectral sequence

$$H^p(X, \underline{\mathrm{Ext}}^q_{\mathcal{O}_X}(\Omega_{X/S}, I)) \Rightarrow \mathrm{Ext}^n_{\mathcal{O}_X}(\mathcal{O}_{X/S}, I)$$

However $\underline{\mathrm{Ext}}^1_{\mathcal{O}_X}(K/K^2, I)$ is frequently not equal to $\underline{\mathrm{Ext}}^2_{\mathcal{O}_X}(\Omega_{X/S}, I)$, even if S = Spec k, where k is a field.

§3. Formal Moduli.

3.1. The category $\underline{C}_{\Lambda}$

Definition 1. Let Λ be a complete noetherian local ring, with maximal ideal μ and residue field $k = \Lambda/\mu$. We define $\underline{C}_{\Lambda}$ to be the category of all local Λ algebras A such that

(i) A is an artin local ring.

(ii) A is a finitely generated Λ module.

(iii) $k = \Lambda/\mu \longrightarrow A/\underline{m}_A$ is a bijection, where $\underline{m}_A$ is the maximal ideal in A.

Morphisms in $\underline{C}_{\Lambda}$ are defined to be local homomorphisms of algebras. We define $\hat{\underline{C}}_{\Lambda}$ to be the category of complete local Λ algebras A for which $A/\underline{m}_A^n \in \underline{C}_{\Lambda}$, all n. We shall write $\underline{C}$ or $\hat{\underline{C}}$ when no confusion can arise. Notice that $\underline{C}$ is a full subcategory of $\hat{\underline{C}}$.

Note that if $A \in \underline{C}$, and M is a finitely generated A module, then for any extension (§1.3) $0 \to M \to E \to A \to 0$ of A/Λ, we have that $E \in \underline{C}$*, and $E \to A$ is a morphism in $\underline{C}$. By an extension of A/Λ we shall always mean one of the above type, where M is a finitely generated A-module. If E and F are extensions of A/Λ,

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[*] More generally, if E is any ring, and J is a nilpotent ideal in E, then the units in E are the inverse images of the units in E/J, and E is local if and only if E/J is local.

and $\alpha : E \longrightarrow F$ is a morphism in $\underline{C}$, we say $\alpha : E \longrightarrow F$ is a morphism over A if α commutes with the projections. (No restriction is placed on the induced map of the kernels.)

If $p : A \longrightarrow B$ is a morphism in $\underline{C}$, and E is an extension of B, we denote the extension $E \underset{B}{\times} A$ of A by $p_*(E)$, as usual.

Definition 2. Let $A \in \underline{C}$ have maximal ideal $\underline{m}$. A small extension E of A/Λ is an extension $0 \longrightarrow I \longrightarrow E \longrightarrow A \longrightarrow 0$ with $\underline{m}I = 0$.

Definition 3. We say that a small extension $p : E \longrightarrow A$ is essential, if whenever $\alpha : F \longrightarrow E$ is a $\underline{C}_{\Lambda}$-morphism such that $p \circ \alpha$ is surjective, then α is surjective.

Since $T^1(A/\Lambda, \cdot)$ is an additive functor, $T^1(A/\Lambda, V) \cong \bigoplus_{i=1}^{q} T^1(A/\Lambda, k)$ for any vector space V of dimension q. Similarly, it is easy to see that any small extension E of A is isomorphic over A to $E_1 \underset{A}{\times} \cdots \underset{A}{\times} E_q$ for some $[E_i] \in T^1(A/\Lambda, k)$, where q = length E - length A. The following guarantees that essential extensions exist in great abundance.

Proposition 1. Let $A \in \underline{C}$ have maximal ideal $\underline{m}$, and let E be a small extension of A with maximal ideal $\underline{n}$. The following are equivalent:

(i) $p : E \longrightarrow A$ is essential.

(ii) $\underline{n}/(\underline{n}^2 + \mu E) \longrightarrow \underline{m}/(\underline{m}^2 + \mu A)$ is a bijection.

$(ii)_{bis}$ Every endomorphism of E (over A) is an isomorphism.

(iii) There exist linearly independent $[E_i]$ in $T^1(A/\Lambda, k)$ such that

$$E \cong E_1 \underset{A}{\times} \cdots \underset{A}{\times} E_q \text{ over } A.$$

$(iii)_{bis}$ In <u>every</u> such decomposition $E \cong E_1 \underset{A}{\times} \cdots \underset{A}{\times} E_q$ over A, the $[E_i]$ are linearly independent in $T^1(A/\Lambda, k)$.

Before proceeding with the proof, we prove a lemma.

<u>Lemma.</u> Let $E = E_1 \underset{A}{\times} \cdots \underset{A}{\times} E_q \xrightarrow{p} A$ be a small extension of A, where $[E_i] \in T^1(A/\Lambda, k)$. Let $V \subseteq T^1(A/\Lambda, k)$ be the subspace spanned by the $[E_i]$. Then

$$V = \ker \left(T^1(A/\Lambda, k) \xrightarrow{p^*} T^1(E/\Lambda, k)\right)$$

<u>Proof.</u> Clearly $V \subseteq \ker p^*$, since the extension $p^*[E_i] = [E \underset{A}{\times} E_i]$ of E is trivial ($pr_1 : E \to E_i$ induces a section). We prove the reverse inclusion

(1) $\ker p^* \subseteq V$

by induction on q = length E - length A.

For $q = 1$, let $[E], [F] \in T^1(A/\Lambda, k)$ and $p : E \to A$ be the projection. $p^*[F] = 0$ means there is a map $\alpha : E \to F$ over A, whose restriction to $k \subset E$ must be multiplication by a scalar. Hence there exists $a \in k$ such that $a[E] = [F]$, i.e., $[F] \in V$.

Now suppose the inclusion (1) is true for all small extensions $F = F_1 \underset{B}{\times} \cdots \underset{B}{\times} F_r \to B$ with $r < q$. Consider the sequence $E = E_1 \underset{A}{\times} \cdots \underset{A}{\times} E_q \xrightarrow{pr_1} E_1 \xrightarrow{p_1} A$. We have $E \cong p_1^*(E_2) \underset{E}{\times} \cdots \underset{E}{\times} p_1^*(E_q)$

over E_1, and $p = p_1 \circ pr_1$. Let $[F] \in T^1(A/\Lambda, k)$ and suppose $p^*[F] = pr_1^* p^*[F] = 0$. Then by the induction hypothesis, applied to pr_1, we know that $p_1^*[F] \in \mathrm{Span}\,(p_1^*[E_2], \ldots, p_1^*[E_q]) \subseteq T^1(E/\Lambda, k)$. But the kernel of p_1^* is spanned by $[E_1]$ (case $q = 1$), so that $[F] \in \mathrm{Span}([E_1], \ldots, [E_q]) = V$, and we are done.

Corollary. In the above situation, let $\underline{m}$ (resp. $\underline{n}$) be the maximal ideal of A (resp. E). Then

$$\dim_k\left(\underline{n}/(\underline{n}^2 + \mu E)\right) - \dim_k\left(\underline{m}/(\underline{m}^2 + \mu A)\right) = q - \dim_k V. \tag{3}$$

Proof. This follows from the exact sequence

$$0 \to \mathrm{Der}_\Lambda(A, k) \to \mathrm{Der}_\Lambda(E, k) \to T^1(A/E, k)$$
$$\to T^1(A/\Lambda, k) \to T^1(E/\Lambda, k)$$

and the equalities

$$\begin{aligned} \mathrm{Der}_\Lambda(A, k) &= \left(\underline{m}/(\underline{m}^2 + \mu A)\right)^* \\ \mathrm{Der}_\Lambda(E, k) &= \left(\underline{n}/(\underline{n}^2 + \mu E)\right)^* \\ T^1(A/E, k) &= J^* \qquad (E/J = A) \end{aligned}$$

where $*$ = vector space dual.

We now proceed to the proof of proposition 1. By the corollary above, we have that

(ii) $\Leftrightarrow$ (iii) $\Leftrightarrow$ (iii)bis.

(ii) $\Rightarrow$ (i). Let $\alpha : F \to E$ be a map over A such that $p \circ \alpha$ is surjective, and let $\underline{r}$ be the maximal ideal of F. The image $\alpha(\underline{r})$ of $\underline{r}$ is an *ideal* in E; in fact if y e $\underline{r}$ and x e E, there exists z e F such that $x - \alpha(z)$ e ker p, whence $\alpha(y)(x - \alpha(z)) = 0$ and $x\alpha(y) = \alpha(yz)$. In particular, $\mu E \subset \alpha(\underline{r})$. Now, by assumption $\underline{r}/(\underline{r}^2 + \mu F) \rightarrow \underline{n}/(\underline{n}^2 + \mu E)$ is surjective, so we have

$$\alpha(\underline{r}) + \mu E + \underline{n}^2 = \alpha(\underline{r}) + \underline{n}^2 = \underline{n}$$

Hence, by Nakyama's lemma, we have $\alpha(\underline{r}) = \underline{n}$. This, together with the fact that $F/\underline{r}F \xrightarrow{\sim} E/\underline{n}E$, implies that α is surjective.

(i) $\Longrightarrow$ (ii)bis. We need only observe that a surjective endomorphism of any small extension is an isomorphism.

(ii)bis $\Longrightarrow$ (iii)bis. Let $E \cong E_1 \underset{A}{x} \dots \underset{A}{x} E_q$ be a decomposition of E ($[E_i]$ e $T^1(A/\Lambda, k)$). Suppose, say, that $[E_1]$ e Span $([E_2], \dots, [E_q]) \subseteq T^1(A/\Lambda, k)$. Then, by the lemma, there exists a morphism $E_2 \underset{A}{x} \dots \underset{A}{x} E_q \longrightarrow E_1$ over A, and hence a morphism $E_2 \underset{A}{x} \dots \underset{A}{x} E_q \longrightarrow E$, over A. Composing this last with the projection $E \longrightarrow E_2 \,x \dots x\, E_q$ we get an endomorphism of E which is clearly not surjective, contrary to hypothesis. <u>Q. E. D.</u>

3.2. Functors on $\underline{C}_\Lambda$.

Let $F : \underline{C}_\Lambda \longrightarrow$ <u>(sets)</u> be a <u>covariant</u> functor. By a <u>couple</u> for F we mean a pair (A, η) where A is in $\underline{C}_\Lambda$ and η e $F(A)$. A <u>morphism of couples</u> (A, η) (B, ξ) consists of a morphism $\alpha : A \longrightarrow B$ in $\underline{C}_\Lambda$ such that $F(\alpha)(\eta) = \xi$. If we extend F to a functor $\hat{F}$ on $\hat{\underline{C}}_\Lambda$ by the formula $\hat{F}(A) = \varprojlim_n F(A/\underline{m}_A^n)$ we may speak analagously of pro-couples (A, ξ) with A e $\hat{\underline{C}}_\Lambda$ and ξ e $\hat{F}(A)$, and morphisms of procouples.

To any ring R in $\hat{\underline{C}}_\Lambda$ corresponds a functor $F_R : A \rightsquigarrow \mathrm{Hom}(R, A)$, for A in $\underline{C}_\Lambda$. On the other hand, if F is any functor from $\underline{C}_\Lambda$ to sets, and R is in $\underline{C}_\Lambda$, we have a bijection : $\mathrm{Hom}(F_R, F) \xrightarrow{\sim} \varprojlim_n F(R/\underline{m}_R^n) = \hat{F}(R)$; namely, if $\pi_n : R \to R/\underline{m}^n = R_n$, is the projection, then $\pi_n \in F_R(R_n)$ and we associate to each $\theta : F_R \to F$ the element $\varprojlim \theta(\pi_n) \in \hat{F}(R)$. We therefore say that a procouple (R, ξ) for F *pro-represents* F if the morphism $F_R \to F$ determined by ξ is an isomorphism.

The "formal moduli" functors which we wish to consider are in general not pro-representable but may be "approximated" in the following sense:

Definition 1. Let $F : \underline{C}_\Lambda \to \underline{\mathrm{Sets}}$ be a covariant functor, and let (R, ξ) be a procouple for F. We say that (R, ξ) is a *projective Hull* of F if the following hold:

(H_1): For any A in $\underline{C}$ with $\underline{m}_A^2 = (0)$, we have $\mathrm{Hom}(R, A) \xrightarrow{\sim} F(A)$.

(H_2): Let $p : (A, \eta) \to (\bar{A}, \bar{\eta})$ be a morphism of couples where $p : A \to \bar{A}$ is a surjection in $\underline{C}_\Lambda$. Then any morphism $\alpha : (R, \xi) \to (\bar{A}, \bar{\eta})$ lifts to (at least one) morphism $\beta : (R, \xi) \to (A, \eta)$ such that $p \circ \beta = \alpha$

$$\begin{tikzcd}
(R, \xi) \arrow[r, dashed, "\exists\ \beta"] \arrow[dr, "\alpha"'] & (A, \eta) \arrow[d, "p\ :\ \text{surjective}"] \\
 & (\bar{A}, \bar{\eta})
\end{tikzcd}$$

commutative.

Remark 1. It follows from the definition that if (R, ξ) is a projective hull of F, then the morphism $\mathrm{Hom}(R, A) \to \hat{F}(A)$ is surjective for any

A in $\hat{\underline{C}}$. However we see no reason why the condition (H_2) will also be satisfied in the larger category of pro-couples (A, η). (Unless (R, ξ) actually pro-represents F.)

Remark 2. It is clear that if (R, ξ) pro-represents F then (R, ξ) is a projective hull of F.

Remark 3. If (R, ξ) is a projective hull of F, then R is formally simple over Λ (§1.2) if and only if F transforms surjections $A \longrightarrow B$ in $\underline{C}$ into surjections $F(A) \longrightarrow F(B)$.

Projective hulls are unique up to non canonical isomorphism. More precisely

Proposition 2. Let $F : \underline{C} \longrightarrow \underline{(\text{Sets})}$ be a functor, and let (R, ξ) and (S, η) be projective hulls of F. Then there exists an isomorphism $\alpha : R \longrightarrow S$ such that $\hat{F}(\alpha)(\xi) = \eta$.

Proof. Let $\underline{m} = \underline{m}_R$ and $\underline{n} = \underline{m}_S$. Let $R_k = R/\underline{m}^k$, $S_k = S/\underline{n}^k$ and let ξ_k, η_k be the induced elements in $F(R_k)$ (resp. $F(S_k)$). By (H_1) there exists a unique morphism $\alpha_2 : R \longrightarrow S_2$ such that $\hat{F}(\alpha_2)(\xi) = \eta_2$. ($\alpha_2$ therefore induces $\bar{\alpha}_2 : R_2 \hookrightarrow S_2$). Suppose we have found $\alpha_k : R \longrightarrow S_k$ with $\hat{F}(\alpha_k)(\xi) = \eta_k$. Then by (H_2) there exists $\alpha_{k+1} : R \longrightarrow S_{k+1}$ such that $\hat{F}(\alpha_{k+1})(\xi) = \eta_{k+1}$. Thus by induction we get an $\alpha : R \longrightarrow S$ with $\hat{F}(\alpha)(\xi) = \eta$, and α induces $R_2 \hookrightarrow S_2$. Taking a similar map $\beta : S \longrightarrow R$ we claim that $\alpha \circ \beta$ and $\beta \circ \alpha$ are isomorphisms. In fact any local endomorphism θ of R which leaves $R/\underline{m}^2$ fixed is an isomorphism (θ is a surjection, since it induces a surjection on the graded rings. On the other hand, the commutative diagram

$$\begin{tikzcd}
0 \arrow[d] & 0 \arrow[d] & \\
\underline{m}^n/\underline{m}^{n+1} \arrow[r] \arrow[d] & \underline{m}^n/\underline{m}^{n+1} \arrow[d] & \\
R_{n+1} \arrow[r, "\theta_{n+1}"] \arrow[d] & R_{n+1} \arrow[r] \arrow[d] & 0 \\
R_n \arrow[r, "\theta_n"] \arrow[d] & R_n \arrow[r] \arrow[d] & 0 \\
0 & 0 &
\end{tikzcd}$$

may be chased to show that if θ_n is an isomorphism, then θ_{n+1} is an isomorphism.) Thus $\alpha \circ \beta$ and $\beta \circ \alpha$ are isomorphisms, leaving η and ξ fixed, and we are done.

We shall see that a necessary and sufficient condition that F have a projective hull (resp. be pro-representable) is that F be half exact (resp. exact) in the following sense.

Definition. Let $F : \underline{C}_\Lambda \longrightarrow \underline{\text{Sets}}$ be a covariant functor. We say that F is half exact if it has the following properties

(E_1) $F(k) = (e)$ (= one point.)

(E_2) : For any morphism $A \longrightarrow B$ in $\underline{C}_\Lambda$ and any small extension $C \longrightarrow B$ the map

$$(*) \qquad F(A \underset{B}{\times} C) \longrightarrow F(A) \underset{F(B)}{\times} F(C)$$

is surjective, and is bijective if $B = k$.

(E_3) The vector space $F(D)$ has finite dimension (where $D = k[\varepsilon]/\varepsilon^2$ is the algebra of dual numbers over k).

If in addition, the map (*) in (E_2) is *always* a bijection (for any B in $\underline{C}$) we say that F is *exact.*

(The vector space structure on $F(D)$ comes from (E_2) and Proposition 2 of §2. 1.)

It is clear that a pro-representable functor is exact.

Half exact functors have the following "additive" property.

Lemma 2. Let F be a half exact functor from $\underline{C}$ to $(\underline{\text{Sets}})$. Let (A, ξ) be a couple for F, and let V_ξ be the subset of $T^1(A/\Lambda, k)$ consisting of the classes of those extensions $E \xrightarrow{p} A$ for which $\xi \in$ image $F(p)$. Then V_ξ is a vector subspace of $T^1(A/\Lambda, k)$.

Proof. Let $[E_1], [E_2] \in V_\xi$. Then the sum E may be included in a sequence $E_1 \underset{A}{\times} E_2 \longrightarrow E \longrightarrow A$. Hence by condition (E_2), V_ξ is closed to addition. If $[E] \in V_\xi$ and $a \in k$, there is a morphism $E \longrightarrow aE$ over A. Hence V_ξ is a vector space.

We now prove

Theorem 1. Let F be a functor from $\underline{C}_\Lambda$ to $(\underline{\text{sets}})$. Then the following are equivalent:

(i) F has a projective hull (R, ξ).

(ii) F is half exact.

Proof (i) $\Longrightarrow$ (ii). Let (R, ξ) be a projective hull of F. Then it follows immediately that F satisfies (E_1) and (E_3). To prove (E_2) let

$u : (A, \zeta) \longrightarrow (B, \eta)$ and $v : (C, \theta) \longrightarrow (B, \eta)$ be morphisms of couples, where v is a small extension. Then there exists $\alpha : (R, \xi) \longrightarrow (A, \zeta)$, and hence by (H_2), there exists $\beta : (R, \xi) \longrightarrow (C, \theta)$ with $u \circ \alpha = v \circ \beta$.

$$\begin{tikzcd}
 & (A \underset{B}{\times} C, \tau) \arrow[r, "pr_2"] \arrow[dd, "pr_1"'] & (C, \theta) \arrow[dd, "v\quad\text{small}"] \\
(R, \xi) \arrow[ur, dashed, "\alpha \times \beta"] \arrow[urr, "\beta"'] \arrow[dr, "\alpha"'] & & \\
 & (A, \zeta) \arrow[r, "u"'] & (B, \eta)
\end{tikzcd}$$

Hence $\tau = F(\alpha \times \beta)(\xi) \in F(A \underset{B}{\times} C)$ projects onto ζ and θ. Suppose $B = k$ (whence $\eta = e$). If τ' is another element of $F(A \underset{B}{\times} C)$ projecting onto ζ and θ, then there is a morphism $\alpha \times \beta' : (R, \xi) \longrightarrow (A \underset{k}{\times} C, \tau')$ lifting α (by (H_2) applied to pr_1). But by (H_1), $\beta' = \beta$. Hence $\tau' = \tau$.

<u>Q. E. D.</u>

(ii) $\Longrightarrow$ (i). Suppose F is half exact. Let $t_1, \ldots, t_m$ be a basis of $F(D)^*$ ($* =$ dual) and set $R_2 = k[t_1, \ldots, t_m]/\underline{t}^2$ ($= D \underset{k}{\times} \cdots \underset{k}{\times} D$ m times).

Let $\eta_i \in F(D)$ be dual to t_i. Then by (E_2) there exists a unique $\xi = \xi_2 \in F(R_2)$ which projects onto each η_i.

Suppose we have found R_n and ξ_n. Then (R_{n+1}, ξ_{n+1}) is constructed as follows. Let $[E_1], \ldots, [E_q]$ be a basis for $V_{\xi_n} \subseteq T^1(R_n/\Lambda, k)$. If we put $R_{n+1} = E_1 \underset{R_n}{\times} \cdots \underset{R_n}{\times} E_q$, then by (E_2) there exists $\xi_{n+1} \in F(R_{n+1})$ which projects onto ξ_n.

Now by Proposition 1 of §3.1, each extension $R_{n+1} \longrightarrow R_n$ is essential. Hence there exist ideals J_n in $S = \Lambda[[t_1, \ldots, t_m]]$ such that

$R_n = S/J_n$ and $R = \varprojlim R_n = S/\bigcap J_n$ is in $\hat{\underline{C}}_\Lambda$. Set $\xi = \varprojlim \xi_n \in \hat{F}(R)$.

We claim that (R, ξ) is a projective hull of F. First, we know that $F_R(D) \xrightarrow{\sim} F(D)$ by choice of R_2. Hence by (E_2), $F_R(A) \xrightarrow{\sim} F(A)$ if $A = D \times_k \ldots \times_k D$ (any number of times). Now to verify (H_2) we may assume that $p : A \longrightarrow \bar{A}$ is a small extension with kernel J. There is an isomorphism

(**)
$$A \times_{\bar{A}} A \xrightarrow{\sim} A \times_k D_k(J) .$$

defined by $(x, y) \longrightarrow (x, x_0 \oplus (y - x))$, where x_0 is the k residue of x.

The morphism $\alpha : (R, \xi) \longrightarrow (\bar{A}, \bar{\eta})$ factors through $(R_n, \xi_n) \longrightarrow (\bar{A}, \bar{\eta})$ for some n. By (E_2) there exists $\zeta \in F(R_n \times_{\bar{A}} A)$ such that $F(pr_1)(\zeta) = \xi_n$ and $F(pr_2)(\zeta) = \eta$.

$$\begin{tikzcd}
u : R_{n+1} \arrow[r, dashed, "\exists"] \arrow[dr, "\pi"'] & R_n \times_{\bar{A}} A \arrow[d, "pr_1"] \\
& R_n
\end{tikzcd}$$

Thus by the choice of R_{n+1}, and Lemma 1 of §3.1 $\pi^*[R_n \times_{\bar{A}} A] = 0$ in $T^1(R_{n+1}/\Lambda, k)$; i.e., there exists $u : R_{n+1} \longrightarrow R_n \times_{\bar{A}} A$ over R_n. Let $\beta' : R \longrightarrow A$ be the map induced by composing u with pr_2; we have $p \circ \beta' = \alpha$.

Now, using the isomorphism (**) and (E_2) we get a commutative diagram

$$\begin{tikzcd}
F_R(A) \times F_R(D_k(J)) \arrow[r] \arrow[d, "\cong"] & F(A) \times F(D(J)) \arrow[d, "r"] \\
F_R(A) \underset{F_R(\bar{A})}{\times} F_R(A) \arrow[r] & F(A) \underset{F(\bar{A})}{\times} F(A)
\end{tikzcd}$$

where r is a surjection. Let $G = F_R(D_R(J))$ $(\hookrightarrow F(D_k(J)) \hookrightarrow F(D) \underset{k}{\otimes} J.)$ The diagram above may be regarded as giving a group action of G on the set $F_R(p)^{-1}(\alpha)$ (resp. $F(p)^{-1}(\bar{\eta})$). $F_R(p)^{-1}(\alpha)$ is a formally principal homogeneous space under G, while the action of G on $F(p)^{-1}(\alpha)$ is only known to be "transitive" (the action is free, i.e., $(\eta)^\sigma = \eta$ implies $\sigma = 0$, if and only if $F(A \underset{\bar{A}}{\times} A) \longrightarrow F(A) \underset{F(\bar{A})}{\times} F(A)$ is bijective.) Now consider $\beta' \in F_R(p^{-1})(\alpha)$ and set $F(\beta')(\xi) = \eta' \in F(p)^{-1}(\bar{\eta})$. Since the action of G is transitive, there exists $\sigma \in G$ such that $(\eta')^\sigma = \eta$. Hence if $\beta = (\beta')^\sigma$, $F(\beta)(\xi) = \eta$, $\beta \in F_R(p)^{-1}(\alpha)$. Thus condition (H_2) is verified.

Finally, to verify (H_1), let $0 \to J \to A \to k$ be a square zero extension of k over Λ, and let $G = F(D_k(J))$. By the above remarks, we know that $F_R(A) \to F(A)$ is a morphism of formally principal G spaces and that one is void if and only if the other is. Hence $F_R(A) \xrightarrow{\sim} F(A)$. Q.E.D.

Corollary 1. The projective hull (R, ξ) pro-represents F if and only if for each small extension $0 \to J \to A \xrightarrow{p} \bar{A} \to 0$ in $\underline{C}_\Lambda$, and each $\bar{\eta} \in F(\bar{A})$, the group $G = F(D_k(J))$ acts freely on $F(p)^{-1}(\bar{\eta})$ (provided $F(p)^{-1}(\bar{\eta})$ is not empty).

In fact if G acts freely, then the morphism $\beta : (R, \xi) \longrightarrow (A, \eta)$ lifting $\alpha : (R, \xi) \longrightarrow (\overline{A}, \overline{\eta})$ found above would be unique; the corollary then follows immediately.

Corollary 2. A covariant functor is pro-representable if and only if it is exact.

Corollary 3. If (R, ξ) is a projective hull of F then there is a canonical isomorphism

$$Z_{R/\Lambda} \xrightarrow{\sim} F(D)$$

of vector spaces, where, for each $A \in \hat{\underline{C}}_\Lambda$, we denote by $Z_{A/\Lambda}$ the Zariski tangent space of A over Λ:

$$Z_{A/\Lambda} = \left(\underline{m}/\underline{m}^2 + \mu A\right)^* \qquad \underline{m} = \underline{m}_A .$$

§3. 3. Formal Moduli.

We fix the following data. $\underline{C}_\Lambda$ is the category of artin local Λ algebras with residue field $k = \Lambda/\mu$, and X is a scheme proper[*] over k. Throughout this section we write T^i for $T^i(X/k, \mathcal{O}_X)$. We continue to denote Zariski tangent spaces by "$Z_{B/A}$" (see Cor. 3 to Thm. 1, §3. 2).

3. 3. 1. We define functors F and G from $\underline{C}_\Lambda$ to sets by

(1) $F(A) = \mathrm{Def}(X/k, \operatorname{Spec} A)$

$A \in \underline{C}_\Lambda$.

(2) $G(A) = H^0(X, \underline{\mathrm{Def}}\,(X/k, \operatorname{Spec} A))$

(See §2. 3 for Def and $\underline{\mathrm{Def}}$.)

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[*] This assumption is used only to insure that the various functors involved satisfy the finiteness condition (E_3).

We call F the formal moduli functor of X/k, and G the formal local moduli functor of X/k. (Note that if X is simple over k, then G is the trivial functor: $G(A) = (e)$ for all $A \in \underline{C}_\Lambda$.) If Y is a deformation of X/k to A we denote its class in $F(A)$ by $[Y]$.

The "local class" maps define a morphism of functors $\ell : F \longrightarrow G$, where for each A and each $[Y] \in F(A)$, $\ell[Y]$ is the section of $H^0(X, \underline{\mathrm{Def}}\,(X/k, \operatorname{Spec} A))$ which determines the local isomorphism class of Y.

We claim that F and G are half exact. Clearly, (E_1) is satisfied. For (E_2), consider first the global case (F). Let $A \longrightarrow B$ and $C \longrightarrow B$ be morphisms in $\underline{C}_\Lambda$, where $C \longrightarrow B$ is a small extension, and let $[Y] \in F(A)$ and $[W] \in F(C)$ have the same projection $[Z]$ in $F(B)$. If $B = k$, then $Z = X$ and we may apply §2.3, p. 28 directly (here $S = \operatorname{Spec} k$, $T_1 = \operatorname{Spec} A$, $T_2 = \operatorname{Spec} C$). For general B, we pick a $Y \in [Y]$, and choose $Z = Y \times_{\operatorname{Spec} A} \operatorname{Spec} B$. Then we choose W to be a deformation of Z/B to C. i.e., we can find a $W \in [W]$ and a closed immersion $Z \hookleftarrow W$ such that the composition $Z \hookleftarrow W \hookleftarrow X$ is the closed immersion defining Z. By Proposition 1 of §1.2, $V = Y \coprod_Z W$ is a deformation of $Y/\operatorname{Spec} A$ to $\operatorname{Spec}(A \times_B C)$ and the morphism $W \longrightarrow V$ induces $W \overset{\hookleftarrow}{\longrightarrow} V \times_{\operatorname{Spec} A \times_B C} \operatorname{Spec} C$. (See the diagram in Proposition 1, §1.2). In particular $[V] \in F[A \times_B C]$ projects on to $[Y] \in F(A)$ and $[W] \in F(C)$. Thus (E_2) is verified. A similar argument works in the local case. (E_3) then follows from Thm. 1, §2, 3, p. 32.

Note: We cannot apply p. 28 of §2.3 to the case of arbitrary B and conclude that $F(A \times_B C) \longrightarrow F(A) \times_{F(B)} F(C)$ is always a bijection, even if $A \longrightarrow B$

is a small extension. For example there may exist deformations Z, Z' of $W/\mathrm{Spec}\, B$ to $\mathrm{Spec}\, C$ such that $[Z] = [Z']$ in $F(C)$, and yet Z is not isomorphic to Z' as a deformation of $W/\mathrm{Spec}\, B$ to $\mathrm{Spec}\, C$. Then we might well have $[Y \underset{W}{\amalg} Z] \neq [Y \underset{W}{\amalg} Z']$. This possibility will be examined in more detail below.

By Theorem 1 of §3. 2 we conclude that F and G have projective hulls (R, ξ) and (T, η). The object $\xi = \varprojlim \xi_n$ is represented by a formal scheme $\mathfrak{X} = \varinjlim X_n$, where X_n is a deformation of X/k to R_n. $\mathfrak{X}$ could be regarded as the generic (infinitesmal) deformation of X and X_2 as the generic linear* deformation of X.

Now by the Corollary 2 to Theorem 1 we have canonical isomorphisms

$$Z_{T/\Lambda} \xrightarrow{\sim} H^0(X, T^1(X/k, \mathcal{O}_X))$$

and

$$Z_{R/\Lambda} \xrightarrow{\sim} \mathrm{Def}(X/k, \overset{\mathrm{Spec}\ D}{\cancel{\mathcal{O}_X}})$$

On the other hand the morphism $\ell : F \longrightarrow G$ induces (at least one) morphism $\alpha : T \longrightarrow R$ such that $G(\alpha)(\eta) = \ell(\xi)$. α must induce a unique morphism $\alpha_2 : T_2 \longrightarrow R_2$ (since both functors are exact (i.e. representable) when restricted to the category of square zero extensions of Λ over k; in fact (R_2, ξ_2) represents (in the sense of Grothendieck) the morphism $\ell_2 : F^{(2)} \longrightarrow G^{(2)}$ of functors on the category $\underline{C}_T^{(2)}$ of square zero extensions of k over T. (Notation obvious.)

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*By a linear deformation we mean a deformation of X/k to A, where $\underline{m}_A^2 = (0)$.

Thus, it is clear that $Z_{R/T} = Z_{R_2/T_2} \hookrightarrow H^1(X, T^o(X/k, \mathcal{O}_X))$, .
and we have an exact sequence

$$(*) \qquad 0 \longrightarrow Z_{R/T} \longrightarrow Z_{R/\Lambda} \xrightarrow{\ell_0} Z_{T/\Lambda}$$

which is the same as (q) of §2.3. Note that

$$\dim_k Z^{\ell}_{T/\Lambda} \leq \dim_k H^o(X, T^1) + \dim_k H^1(X, T^o)$$

$(T^i = T^i(X/k, \mathcal{O}_X))$ with equality if and only if the map $\rho : H^o(X, T^1) \longrightarrow H^2(X, T^o)$ is zero, or, if and only if every germ of a linear deformation comes from a global linear deformation, i.e., if and only if ℓ_0 in (*) is surjective.

Remark. The morphism $\ell : F \longrightarrow G$ need not be "relatively representable" or even "approximable" in the sense that F and G are approximable by their respective projective hulls. For example, let (A, η) be a pro-couple for G. For any morphism $u : A \longrightarrow B$, put $\eta_B = G(u)(\eta)$. Then define F_η on the category $\underline{C}_A$ by setting $F_\eta(B) = \ell^{-1}(\eta_B)$. F_η is not, in general, half exact. One can easily see that this will be the case when F is exact, but G is not, and R is "simple over T". For example the curve $zy^2 = x^2z + x^3$ in projective coordinates x, y, z has T = R; F is exact, but G is not.

3.3.2. Simplicity of the projective hull.

Recall that a morphism $A \longrightarrow B$ of complete local rings with the same residue fields is formally simple if and only if B is a formal power series ring over A. (This is a simple consequence of the definition (§1.3) of formal simplicity.)

Let $0 \to J \to A_1 \to A_0 \to 0$ be a small extension in $\underline{C}_\Lambda$, and let X_0 be a deformation of X/k to A_0. Then by the results of §2, we know that $\mathrm{Ex}(X_0/\mathrm{Spec}\ A_0,\ J \otimes \mathcal{O}_{X_0}) \cong \mathrm{Ex}(X/k,\ \mathcal{O}_X) \otimes_k J$, $T^1(X_0/\mathrm{Spec}\ A_0,\ J \otimes \mathcal{O}_X) \simeq T^1 \otimes_k J$. There are sequences (pg 28)

(1) $0 \to H^{\phi}(X,\ T^0) \otimes J \to \mathrm{Ex}(X/k,\ \mathcal{O}_X) \otimes J \xrightarrow{\ell \otimes J} H^{\overset{0}{1}}(X,\ T^1) \otimes J$

$\xrightarrow{\rho \otimes J} H^{\overset{2}{0}}(X,\ T^{\overset{0}{2}}) \otimes J$ (exact)

and

(2) $\mathrm{Def}(X_0/A_0,\ A_1) \to H^0(X,\ \underline{\mathrm{Def}}(X_0/A_0,\ A_1)) \to H^2(X,\ T^{\overset{0}{2}}) \otimes J$.

and we have obstructions $c^{0,2} \in H^{\overset{0}{2}}(X,\ T^{\overset{2}{0}}) \otimes J$, $c^{1,1} \in H^1(X,\ T^1) \otimes J$ (if $c^{0,2} = 0$), and $c^{2,0}(\eta_1)$ for $\eta_1 \in H^0(X,\ \underline{\mathrm{Def}}(X_0/A_0,\ A_1))$ (if $c^{0,2}$ and $c^{1,1} = 0$).

Thus R is formally simple over Λ if and only if for each small extension $A_1 \to A_0$ and each $[X_0] \in F(A_0)$, $c^{1,1} = c^{0,2} = 0$, and $c^{2,0}(\eta_1) = 0$ for some η_1.

Now for the simplicity of "R over T" we have

<u>Proposition.</u> The following are equivalent

(i) All obstructions $c^{2,0}(\eta_1)$ vanish.

(ii) For some (and hence for all) morphisms $\alpha : T \to R$ (such that $G(\alpha)(\eta) = \ell(\xi)$), R is formally simple over T.

<u>Proof.</u> If for some $\alpha : T \to R$, α is simple, then it is clear that all obstructions $c^{2,0} = 0$. Suppose, conversely that $c^{2,0}(\eta_1) = 0$ for all

η_1. Let $\alpha : T \to R$ be a morphism as described and consider a diagram

$$\begin{tikzcd}
(R, \xi) \arrow[r, dashed] \arrow[dr, "\beta_0" near start] & (A_1, \eta_1, ?) \arrow[d, "p"] & \ell(?) = \eta_1 \\
(T, \eta) \arrow[u, "\alpha"] \arrow[ur, "\alpha_1" near end] \arrow[r, "\alpha_0"'] & (A_0, \eta_0, \xi_0) & \ell(\xi_0) = \eta_0
\end{tikzcd}$$

where all the solid arrows are given data, and all triangles commute. Now the obstruction to finding a $\xi_1 \in F(A_1)$ is, by assumption zero, so we may replace the question mark with a ξ_1 such that $\ell(\xi_1) = \eta_1$. Hence there exists a $\underline{\Lambda}$ homomorphism $\beta_1' : (R, \xi) \to (A_1, \xi_1)$ such that $p \circ \beta_1' = \beta_0$. Let $\alpha_1' = \beta_1' \circ \alpha$. Then $p \circ \alpha_1' = \alpha_0$, hence there exists $\sigma \in H^0(X, T^1) \otimes J$ $(= G(D_k(J))$ such that $(\alpha_1')^\sigma = \alpha_1$. But by assumption again, there exists $\tau \in F(D_k(J))$ such that $\ell \otimes J(\underset{\tau}{\sigma}) = \underset{\sigma}{\tau}$ (see (1) above).

Thus if we put $\beta_1 = (\beta_1')^\tau$ we find that $\beta_1' \circ \alpha = \alpha_1$, i.e., β_1 is a $\underline{T}$ morphism; hence α is a simple morphism.

§3.3.3. Obstructions to representing F.

By the corollary to Theorem 1, §3.2 we know that the obstruction to representing F lies in the possibility that for some small extension $0 \to J \to A_1 \xrightarrow{p} A_0 \to 0$, and some $[X_0] \in F(A_0)$ the operation of $G = F(D_k(J))$ on $F(p)^{-1}([X_0])$ may not be "free". We interpret this condition geometrically.

<u>Lemma.</u> In the above situation, G operates freely on $F(p)^{-1}([X_0])$

(assuming it is not empty) if and only if for every (resp. some) deformation X_1 of X_0/A_0 to A_1, every automorphism θ_0 of X_0/A_0 (restricting to the identity on X/k) extends to an automorphism of X_1/A_1.

Proof. Let us first look at the action of G. If $[X_1] \in F(p)^{-1}[X_0]$, then there is a commutative diagram

$$\begin{tikzcd}
X \arrow[r, hook, "i"] \arrow[d] & X_0 \arrow[r, hook, "j"] \arrow[d] & X_1 \arrow[d] \\
\mathrm{Spec}\, k \arrow[r, hook] & \mathrm{Spec}\, A_0 \arrow[r, hook] & \mathrm{Spec}\, A_1
\end{tikzcd}$$

in which all squares are product diagrams. If $\sigma \in G$, and $[X_1]^{\sigma} = [X_1']$, then we get a large commutative diagram

$$\begin{tikzcd}
& & & X_1' \arrow[ddl] \\
X \arrow[r, hook, "i"] \arrow[d] & X_0 \arrow[r, hook, "j"] \arrow[d] \arrow[urr, hook, bend left, "j'"] & X_1 \arrow[d] \arrow[ur, dashed, "\alpha"] & \\
\mathrm{Spec}\, k \arrow[r, hook] & \mathrm{Spec}\, A_0 \arrow[r, hook] & \mathrm{Spec}\, A_1 &
\end{tikzcd}$$

To say that $[X_1] = [X_1']$ in $F(A_1)$ is to say that there is a morphism $\alpha : X_1 \longrightarrow X_1'$ over Spec A_1 such that $\alpha \circ i \circ j = i \circ j'$. Thus α induces an automorphism $\theta_0 = (\mathrm{id}_{X_0} \underset{A_1}{\times} \alpha) : X_0 \xrightarrow{\sim} X_0$, with $j' \circ \theta_0 = \alpha \circ j$. Now $X_1 \xrightarrow{\sim} X_1'$ as deformations of X_0/A_0 to A_1 (i.e., $\sigma = 0$) if and only if there exists a $\phi : X_1' \xrightarrow{\sim} X_1$ over A_1, such that $\phi \circ j' = j$. If such a ϕ existed, then $\theta_1 = \phi \circ \alpha$ would be an automorphism of X_1/A_1 such that $\theta_1 \circ j = \theta_0 \circ j$, i.e., θ_1 would extend θ_0. Conversely, if θ_1 exists, set $\phi = \theta_1 \circ \alpha^{-1}$.

On the other hand, let θ_0 be an automorphism of X_0/A_0, and let $j : X_0 \hookrightarrow X_1$ be a deformation of X_0/A_0 to A_1. If we put $j' = j \circ \theta_0$, then j' defines a new deformation X'_1 of X_0/A_0 to A_1. It is clear that as deformations of X/k to A_1, $[X_1] = [X'_1]$, while $X_1 \hookrightarrow X'_1$ as deformations of X_0/A_0 to A_1 if and only if θ_0 extends to an automorphism of X_1/A_1. Thus, if some automorphism of X_0/A_0 fails to extend to some deformation X_1 of X_0/A_0 to A_1, then G has a fixed point, namely $[X_1]$.

Finally, to prove the remark in parentheses suppose θ_0 is an automorphism of X_0/A_0, that X_1 and X'_1 are deformations of X_0/A_0 to A_1, and that θ_0 extends to an automorphism θ_1 of X_1/A_1. We claim that θ_0 also extends to an automorphism θ'_1 of X'_1/A_1. In fact such an automorphism θ'_1 is simply an automorphism of $X_1 \amalg_{X_0} X'_1$ over $A_1 \times_{A_0} A_1$ which induces θ_1 on X_1 via the projection $pr_1 : A_1 \times_{A_0} A_1 \to A_1$. On the other hand there exists a deformation Y of X/k to $D_k(J)$ (i.e., $[Y] \in F(D_k(J))$ such that

$$\begin{tikzcd}
X_1 \amalg_{X} Y \arrow[r, hook] \arrow[d] & X_1 \amalg_{X_0} X'_1 \arrow[d] \\
\operatorname{Spec}(A_1 \times_{k} D_k(J)) \arrow[r, hook, "u"'] & \operatorname{Spec}(A_1 \times_{A_0} A_1)
\end{tikzcd}$$

is a product diagram, where u is deduced from the canonical isomorphism $\alpha : A_1 \times_{A_0} A_1 \hookrightarrow A_1 \times_k D_k(J)$ (see proof of Theorem 1, p.44). Since, by definition of α, the diagram

$$\begin{tikzcd}
\alpha : A_1 \underset{A_0}{\times} A_1 \arrow[r] \arrow[d, "pr_1"'] & A_1 \underset{k}{\times} D_k(J) \arrow[d, "pr_1"] \\
A_1 \arrow[r] & A_1
\end{tikzcd}$$

is commutative, we need only find an automorphism ϕ of $X_1 \underset{X}{\amalg} Y$ over $A_1 \underset{k}{\times} D_k(J)$ restricting to θ_1 via $pr_1 : A_1 \underset{k}{\times} D_k(J) \longrightarrow A_1$. Clearly $\phi = (\theta_1 \amalg \mathrm{id}\ Y)$ will do.

Now we may "functorialize" the obstructions to pro-representing F by the following procedure. Let (R, ξ) be a projective hull of F, and choose, once and for all, a fixed representative $\mathfrak{X}$ of ξ where $\mathfrak{X}$ is a formal scheme over R, $\mathfrak{X} = \varinjlim X_n$, with $[X_n] \in F(R_n)$. Let $\underline{C}_R$ be the category of artin local R algebras, of finite type (as modules) over R, and having residue field k.

If $A \in \underline{C}_R$, denote by $\mathfrak{X}_A$ the scheme $X_n \underset{\mathrm{Spec}\, R_n}{\times} \mathrm{Spec}\, A$ over Spec A, where n is any integer such that $R \longrightarrow A$ factors through $R_n \longrightarrow A$. Then define $H : \underline{C}_R \longrightarrow \underline{(\mathrm{Sets})}$ by the formula

$$H(A) = \mathrm{Aut}_A(\mathfrak{X}_A/X) .$$

(= those automorphisms of $\mathfrak{X}_A$ over Spec A which reduce to the identity on X.) By the definition of fibred direct sum, H is an <u>exact</u> functor on the category $\underline{C}_R$. Let (S, θ) pro-represent H. Then θ is an automorphism of $\mathfrak{X}_S = \varinjlim \mathfrak{X}_{S_n}$ over S (we could say that θ is the generic automorphism of $\mathfrak{X}$) and S is a formal lie group over R. (i.e., S is a group object in the category dual to $\hat{\underline{C}}_R$). We find, by the results of §2.3 that $Z_{S/R} \hookrightarrow H^o(X, T^o)$. Furthermore, by the lemma preceding, we know that *S is formally simple over R if and only if $(R, \mathfrak{S})$ pro-represents F.* This is in particular the case when $H^o(X, T^o) = 0$ $(S = R)$.

Now since S is a formal lie group over R, there exists an "identity" section $S \dashrightarrow R$ of the "structure morphism" $R \rightarrow S$. It follows that the exact sequence

$$0 \rightarrow Z_{S/R} \rightarrow Z_{S/\Lambda} \rightarrow Z_{R/\Lambda} \rightarrow T^1(X/R, k) \rightarrow$$
$$T^1(X/\Lambda, k) \rightarrow T^1(R/\Lambda, k)$$

decomposes into the shorter exact sequences

(a) $\quad 0 \rightarrow Z_{S/R} \rightarrow Z_{S/\Lambda} \rightarrow Z_{R/\Lambda} \rightarrow 0 \quad$ and

(b) $\quad 0 \rightarrow T^1(S/R, k) \rightarrow T^1(S/\Lambda, k) \rightarrow T^1(R/\Lambda, k) \rightarrow 0$

By means of the exact sequences (a) and (b) one can see how to "create" functorially, the obstructions to representing F, which by general theory must lie in $T^1(S/R, k)$. We leave the details to the reader.

3.3.4. Algebracizing $\mathfrak{X}$.

We consider briefly the problem of algebracizing $\mathfrak{X}$, i.e., finding an *ordinary* scheme Y, proper over R, and compatible morphisms $Y \leftarrow X_n$ inducing $Y \underset{R}{\otimes} R_n \overset{\frown}{\longleftrightarrow} X_n$. Suppose $\mathcal{L}_n$ is an invertible sheaf on X_n. Since the homomorphisms of an invertible sheaf are canonically isomorphic to the structure sheaf, it follows that the obstruction to lifting $\mathcal{L}_n$ to an invertible sheaf $\mathcal{L}_{n+1}$ on X_{n+1} lies in $H^2(X, \mathcal{O}_X) \otimes \underline{m}^n/\underline{m}^{n+1}$ and that the set of isomorphism classes of such liftings is a formally principal homogeneous space under $H^1(X, \mathcal{O}_X) \otimes \underline{m}^n/\underline{m}^{n+1}$. By EGA II, we know that $\mathcal{L}_{n+1}$ is ample if and only if $\mathcal{L}_n$ is ample. Thus if X is projective and $\mathcal{L}_0$ an ample sheaf on X, we can find a sequence $(\mathcal{L}_n)$ of ample sheaves on (X_n) if all the obstructions vanish. Then we may apply the fundamental existence theorem of EGA III, to conclude that $\mathfrak{X}$ is algebracizable (in fact Y will be projective over R, with an ample sheaf $\mathcal{L}$ restricting to the $\mathcal{L}_n$.)

We collect these results in

Theorem 2. Let X be a scheme proper* over $k = \Lambda/\mu$. Define covariant functors F and G from $\underline{C}_\Lambda$ to (Sets) by

$$F(A) = \mathrm{Def}(X/k, \operatorname{Spec} A)$$

$$G(A) = H^0(X, \underline{\mathrm{Def}}\,(X/k, \operatorname{Spec} A)) \quad (\S 2.3.1)$$

1. Then F and G have projective hulls (R, ξ) and (T, η) in the sense of §3.2, Definition 1. There is a (non-canonical)morphism $(T, \eta) \rightarrow (R, \mathcal{l}(\xi))$ which is simple if and only if all obstructions $c^{2,0}(\eta) \in H^2(X, T^0) \otimes J$ vanish. (§3.3.2) There are canonical vector space isomorphisms

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*The comment on p. 46 applies throughout.

$$Z_{T/\Lambda} \hookrightarrow H^0(X, T^1), \quad Z_{R/\Lambda} \hookrightarrow Ex(X/k, \mathcal{O}_X)$$

$$Z_{R/T} \hookrightarrow H^1(X, T^0)$$

and an exact sequence

$$0 \longrightarrow Z_{T/\Lambda} \longrightarrow Z_{R/\Lambda} \longrightarrow Z_{R/T} \longrightarrow H^2(X, T^0).$$

2. Choose a representative $\mathfrak{X}$ of $\mathfrak{S}$ as a formal scheme over R. Then there exists a formal lie group S over R which <u>pro-represents</u> the functor $A \rightsquigarrow \mathrm{Aut}_A(\mathfrak{X}_A/X)$ from $\underline{C}_R$ to (<u>Sets</u>). $Z_{S/R} \hookrightarrow H^0(X, T^0)$, and S is formally simple over R if and only if $(R, \mathfrak{S})$ pro-represents F. (§3. 3. 3)

§4. Reduced Curves.

§4.0 Notations.

In this section, we deal only with the equi-characteristic case $\Lambda = k$, where k is a fixed algebraically closed field. We write $\underline{C}$ for $\underline{C}_k$. (§3.1). If X is a scheme over k, we write T^i_X, or just T^i, for $T^i(X/k, \mathcal{O}_X)$ (§2.2). If $A \in \underline{C}$ we say a deformation of X/k to A is trivial if it is isomorphic to $X \underset{\operatorname{Spec} k}{\times} \operatorname{Spec} A$. A similar terminology will be used for germs of deformations. That is, a section h of $H^0(X, \underline{\operatorname{Def}}(X/k, \operatorname{Spec} A))$ is trivial if there exists a cover (U_i) of X such that $h(U_i)$ = the trivial deformation of $X|U_i$ to A. Thus, in particular, if F and G are the formal moduli functors for $X|k$ (§3.3) $F(A) \neq \emptyset$, $G(A) \neq \emptyset$ for each A in $\underline{C}$.

4.1. Reduced schemes.

4.1.1. T^1 and T^2 as $\underline{\operatorname{Ext}}$'s

Let X be a scheme of finite type over k. By definition of T^1 and $T_1 = T_1(X/k, \mathcal{O}_X)$ (§1.1) there is an exact sequence

$$\text{(a)} \qquad 0 \to \underline{\operatorname{Ext}}^1_{\mathcal{O}_X}(\Omega_{X/k}, \mathcal{O}_X) \to T^1 \to \underline{\operatorname{Hom}}_{\mathcal{O}_X}(T_1, \mathcal{O}_X)$$

Suppose that there is a closed immersion $X \hookrightarrow Y$ where Y is simple over k and $\mathcal{O}_X = \mathcal{O}_{Y/I}$. Then again by the definition of T_2 and T^2 we have an exact sequence

$$\text{(b)} \qquad 0 \to T^2 \to \underline{\operatorname{Ext}}^1_{\mathcal{O}_X}(I/I^2, \mathcal{O}_X) \to \underline{\operatorname{Hom}}_{\mathcal{O}_X}(T_2, \mathcal{O}_X)$$

Now if X is reduced, then X is generically simple, and T_1, T_2 are generically zero. ([6]). It follows that

$$(1) \qquad \underline{\mathrm{Ext}}^1_{\mathcal{O}_X}(\Omega_{X/k}, \mathcal{O}_X) \hookrightarrow T^1 \qquad \text{and}$$

$$(2) \qquad T^2 \hookrightarrow \underline{\mathrm{Ext}}^1_{\mathcal{O}_X}(I/I^2, \mathcal{O}_X)$$

if X is reduced.

Remark. To get equation (2), it is not necessary to assume that X may be immersed in a simple scheme Y. In fact, if $A \to B$ is a ring homomorphism, M is a B module, and $B = C/I$ where C is formally simple over A, then it can be shown that $\mathrm{Ext}^i_B(I/I^2, M)$, $i \geq 1$, is independent of the choice of C, up to a canonical isomorphism. Hence $\underline{\mathrm{Ext}}^1_{\mathcal{O}_X}(I/I^2, \mathcal{O}_X)$ exists, even though Y may not, and the equation (2) remains true if X is reduced.

4.1.2. Vanishing of T^2.

If $0 \to J \to A_1 \to A_0 \to 0$ is a small extension in $\underline{C}$, and X_0 is a deformation of X/k to A_0, we know that the local obstruction to lifting X_0/A_0 to an X_1/A_1 is a section of $T^2 \otimes_k J$ (§2.3). Hence it is of interest to discover when $T^2_x = (0)$, for $x \in X$. This is certainly the case when X is locally a complete intersection at x (1.3). Another criterion will be given below. First we need a definition.

Definition. Let A be a ring, and M an A module. We denote the torsion submodule of M by $t_A(M)$; that is $t_A(M)$ is the set of $x \in M$ such that $ax = 0$ for some non-zero divisor $a \in A$. If K is the total ring of quotients of A, then $t_A(M)$ is the kernel of $M \to M \otimes K$. We say that M is torsion free if $t_A(M) = (0)$, and that M is a torsion module if $t_A(M) = M$.

Proposition 1. Let B be a regular local ring, let I be an ideal in B such that $A = B/I$ is <u>reduced</u> and <u>Cohen-Macauley,</u> of codimension d. Put $w = \mathrm{Ext}_B^d(A, B)$. Then there is an isomorphism

$$\mathrm{Ext}_A^1(I/I^2, A) \cong t_A(\mathrm{Tor}_B^{d-2}(w, B))$$

<u>Proof</u>. If M is any B module, then the spectral sequence $\mathrm{Ext}_A^p(\mathrm{Tor}_q^B(M, A), A) \Longrightarrow \mathrm{Ext}_B^n(M, A)$ yields an exact sequence

$$0 \to \mathrm{Ext}_A^1(M \otimes_B A, A) \to \mathrm{Ext}_B^1(M, A) \to \mathrm{Hom}_A(\mathrm{Tor}_1^B(M, A), A)$$

from which it follows easily that

$$\mathrm{Ext}_A^1(M \otimes_B A_1, A) \xrightarrow{\sim} t_A(\mathrm{Ext}_B^1(M, A))$$

(Since A is <u>reduced,</u> we know that $\mathrm{Ext}_A^1(P, Q)$ is a torsion A module for any A modules P, Q). Taking $M = I$ we see that $\mathrm{Ext}_A^1(I/I^2, A) \cong t_A(\mathrm{Ext}_B^1(I, A)) \cong t_A(\mathrm{Ext}_B^2(A, A))$.

On the other hand, since A is Cohen Macauley of codimension d, we know that for any B module M

$$\mathrm{Ext}_B^k(A, M) = (0) \quad \text{for } k > d$$

and

$$\mathrm{Ext}_B^k(A, B) = (0) \quad \text{for } k < d.$$

Thus, if we put $F^k(M) = \mathrm{Ext}_B^{d-k}(A, M)$, $0 \leq k \leq d$, it follows that F^o is right exact, that the F^k form a connected sequence of functors for $k \geq 0$, and that the F^k vanish on projectives for $k > 0$, since $F^k(B) = (0)$ $k > 0$. Therefore, by a standard argument, we find that

$$F^k(M) \cong \mathrm{Tor}_B^k(F^o(B), M) = \mathrm{Tor}_B^k(w, M).$$

Taking $M = A$ and $k = d - 2$, we have our result.

Corollary. If $d \leq 2$, then $T^2 = \mathrm{Ext}^1_A(I/I^2, A) = (0)$. If $d = 0$ or 1, this is obvious. If $d = 2$, then $T^2 = t_A(w \otimes_B A) = t_A(w)$. But by [5], § , Ass w = Ass A; that is $t_A(w) = (0)$.

Thus a reduced, Cohen-Macauley scheme, "Locally of codimension 2" has a vanishing T^2.

4.1.3. Rigid singularities.

Definition. Let X be a reduced scheme over k. We say that an isolated singularity $x \in X$ is rigid if $T^1_x = (0)$.

Thus if $x \in X$ is a rigid singularity, and U is an affine open neighborhood of x, then every deformation of U is trivial, (or, the formal moduli functor for the scheme $Y = \mathrm{Spec}\ \mathcal{O}_x/k$ is pro-represented by k itself.)

The example given below of a rigid singularity is due to Grauert and Kerner in [ ], in the context of complex spaces. However, the algebraic verification that $T^1_x = (0)$ is due essentially to S. Lichtenbaum.

* Theorem. The vertex of the cone over $\mathbb{P}_n \times \mathbb{P}_m$ is rigid, for $n \geq 1$, $m \geq 2$. Here we are assuming that $\mathbb{P}_n \times \mathbb{P}_m$ is given the Segre imbedding in $\mathbb{P}^r$, $r = (n + 1)(m + 1) - 1$.

Recall that if $S = k[t_0, \ldots, t_r]$ is any graded ring, generated over k by S_1, then the cone over $\mathrm{Proj}\ S$ is the affine scheme $\mathrm{Spec}\ S$, and the vertex of the cone is the point corresponding to the irrelevant maximal ideal S_+ in S.

Before proceeding to the proof, we shall need some lemmas.

Let A be a ring, and let $S = A[t_0, \ldots, t_r]$ be any graded homogeneous ring, generated over A by S_1. Let $\Omega_{S/A}$ be the module of Kähler differentials, graded by requiring that dt_i have degree 1. Let $X = \mathrm{Proj}\, S$, $Y = \mathrm{Spec}\, A$, $S_i = S(t_i^{-1})$, $T_i = St_i^{-1}$ (thus S_i is the set of homogeneous elements of degree 0 in T_i). Let $\tilde{\Omega}_{S/A}$ be the $\mathcal{O}_X$ module obtained from the graded module $\Omega_{S/A}$. We now define maps

$$p : \Omega_{X/Y} \to \tilde{\Omega}_{S/A} \qquad \text{and}$$

$$q : \tilde{\Omega}_{S/A} \to \mathcal{O}_X$$

as follows. $p \mid \mathrm{Spec}\, S_i$ is defined by a derivation $D_i : S_i \to (\Omega_{S/A})(t_i^{-1})$, where $D_i\left(\frac{f}{t_i^m}\right) = \frac{t_i^m df - fd(t_i^m)}{t_i^{2m}} \in (\Omega_{S/A})(t_i^{-1})$ if $f \in S_m$. It is easy to see that D_i and D_j agree on $\mathrm{Spec}\, S_i \cap \mathrm{Spec}\, S_j = \mathrm{Spec}\, S(t_i^{-1}t_j^{-1})$ and hence the D_i define a homomorphism p. q is defined by a homomorphism $\Omega_{S/A} \to S$ of graded modules: $\sum f_i dt_i \mapsto \sum f_i t_i$ (this is well defined by Euler's Theorem on homogeneous functions; if $f(X_0, \ldots, X_n)$ is a homogensous form such that $f(t_0, \ldots, t_n) = 0$, then $0 = \sum \frac{\partial f}{\partial t_i} dt_i \mapsto \sum \frac{\partial f}{\partial t_i} t_i = (\deg f) \cdot f(t) = 0$, and relations of the type $0 = \sum \frac{\partial f}{\partial t_i} dt_i$ generate the relations on the dt_i's in $\Omega_{S/A}$).

Lemma 1. The sequence $0 \to \Omega_{X/Y} \to \tilde{\Omega}_{S/A} \to \mathcal{O}_X \to 0$ is exact.

Proof. It is clear that $T_i = S_i[t_i, t_i^{-1}]$ is simple over S_i. Hence the sequence

$$(3) \qquad 0 \to (\Omega_{S_i/A}) \underset{S_i}{\otimes} T_i \to \Omega_{T_i/A} \to \Omega_{T_i/S_i} = 0$$

is exact. It is also easy to see that

$$\Omega_{T_i/A} \cong (\Omega_{S/A}) \underset{S}{\otimes} T_i, \quad \Omega_{T_i/S_i} \cong T_i$$

via $dt_i \rightsquigarrow t_i$, and that these identifications commute with the homomorphism q. Hence the result follows by taking homogeneous components of degree 0 in (3).

Lemma 2. Let A be a local domain with maximal ideal $\underline{m}$, and let M be an A module of finite type. Suppose that depth $A \geq 3$, and that $\mathrm{Supp}(\mathrm{Ext}^1_A(M, A)) = \{\underline{m}\}$ (that is $\underline{m}^n \mathrm{Ext}^1_A(M, A) = (0)$, some n). Then if depth $(\mathrm{Hom}_A(M, A)) \geq 3$, $\mathrm{Ext}^1_A(M, A)$ is (0).

Proof. Let us write $\check{N} = \mathrm{Hom}_A(N, A)$ for any A module N. We note that any reflexive module N ($N = (\check{N})^\vee$) has depth ≥ 2; to see this write $0 \to P \to Q \to \check{N} \to 0$ with Q free, thus $0 \to N \to \check{Q} \to R \to 0$ with $R \subset \check{P}$ torsion free. Now apply Ext $(k, \cdot)$ to this last sequence.

Take a resolution

$$0 \to R \to F \to M \to 0$$

with F free. Then we get exact sequences

$$0 \to \check{M} \to \check{F} \to N \to 0$$

$$0 \to N \to \check{R} \to \mathrm{Ext}^1_A(M, A) \to 0$$

Now $\check{R}$ has depth ≥ 2 by the above remark; this if depth $\check{M} \geq 3$, then depth $N \geq 2$, so depth $(\mathrm{Ext}^1_A(M, A)) \geq 1$. This implies that $\mathrm{Ext}^1_A(M, A) = (0)$, since $\underline{m}^n \cdot \mathrm{Ext}^1_A(M, A) = (0)$ for some n.

Corollary. Let X be an integral scheme, having an isolated singularity at $x \in X$, such that $\operatorname{depth}_x X \geq 3$. If depth $T_x^o \geq 3$ then x is a rigid singularity.

Proof. Put $A = \mathcal{O}_x$, $M = \Omega_{A/k}$, and apply the lemma.

Finally, we recall the following from EGA III, about the relation between the depth and cohomology of coherent sheaves on projective varieties.

Theorem. Let $S = k[t_0, \dots, t_r]$ be a graded k algebra, and N a finitely generated graded S module. Let $\underline{m} = S_+ = \sum_{n>0} S_n$, $A = S_{\underline{m}}$, $T = N_{\underline{m}}$, $X = \operatorname{Proj} S$, $\tilde{M}$ and $\mathcal{O}_X(1)$ the sheaf associated to canonical line bundle. Let

$$\alpha : M \longrightarrow \sum_{n=-\infty}^{\infty} H^o(X, \tilde{M}(k))$$

be the canonical homomorphism. Then

(i) depth $T \geq 1 \Longleftrightarrow \alpha$ is injective.

(ii) depth $T \geq 2 \Longleftrightarrow \alpha$ is bijective.

(iii) depth $T \geq d \Longleftrightarrow \alpha$ is bijective and

$$H^p(X, \tilde{M}(k)) = (0), \text{ for } 0 < p < d-1$$

and all k. $(k \geq 3)$.

Now we return to our example (*). In the notation of the preceding theorem, $X = \mathbb{P}^n \times \mathbb{P}^m$. If we take $N = \operatorname{Hom}_S(\Omega_{S/A}, S)$, then $T = N_{\underline{m}} = T_A^o$. By Lemma 2, we must prove that depth $T \geq 3$. By Lemma 1 we have an exact sequence

$$(1) \qquad 0 \rightarrow \mathcal{O}_X \rightarrow \tilde{M} \rightarrow T_X^o \rightarrow 0.$$

Using the relations $\mathcal{O}_X = \mathcal{O}_{\mathbb{P}^n} \otimes \mathcal{O}_{\mathbb{P}^m}$, $\mathcal{O}_X(1) = \mathcal{O}_{\mathbb{P}^n}(1) \otimes \mathcal{O}_{\mathbb{P}^m}(1)$, the Künneth formula[1], and well known facts about the cohomology of $\mathcal{O}_{\mathbb{P}^n}(k)$ we can show directly that A has depth n + m + 1 (that is, A is Cohen Macauley) for any n and m ≥ 0.

On the other hand, the exact sequence

$$0 \to \mathcal{O}_X \to \mathcal{O}_X(1)^{n+1} \to T^o_{\mathbb{P}^n} \to 0$$

shows that $H^o(\mathbb{P}^n, T^o_{\mathbb{P}^n}(k)) = (0)$ if $k \leq -2$ and $H^1(\mathbb{P}^n, T_{\mathbb{P}^n}(k)) = (0)$ if $k \geq 0$. Since

$$T^o_{\mathbb{P}^n x\, \mathbb{P}^m} = (T^o_{\mathbb{P}^n} \otimes \mathcal{O}_{\mathbb{P}^m}) \oplus (\mathcal{O}_{\mathbb{P}^n} \otimes T^o_{\mathbb{P}^m})$$

it follows that $H^1(\mathbb{P}^n \times \mathbb{P}^m, T^o_{\mathbb{P}^n x \mathbb{P}^m}(k)) = (0)$ for all k if $n \geq 1$, $m \geq 2$ (using the Künneth formula again). Thus by (1) above, $H^1(X, \tilde{M}(k)) = (0)$ for all k. Since T is by definition reflexive, depth T ≥ 2, so that the canonical map α is bijective. Hence, by (iii), depth T ≥ 3, and we are done.

§4.2. Curves.

Let X be a reduced curve over k; that is, a reduced scheme, proper over k, all of whose local rings have dimension 0 or 1. Let F (resp. G) be the formal moduli functor (resp. formal local moduli functor) for $X|k$. We denote the respective projective hulls by (R, ξ) and (T, η) (Theorem 2, §3.5). Since every curve is projective and satisfies $H^2(X, \mathcal{O}_X) = (0)$, we know by §3.4 that ξ is represented by the formal completion $\hat{Y}$ of a projective scheme ("curve") Y over R. Y is called the generic

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[1] The Künneth formula states that if V and W are varieties, $\mathcal{F}$ is a coherent sheaf on V and G is a coherent sheaf on W then

$$H^q(V x W, \mathcal{F} \otimes G) \cong \sum_{r+s=q} H^r(V, \mathcal{F}) \otimes H^s(W, G).$$

deformation of X/k.

Since $H^2(X, T^o) = (0)$, the sequence

$$0 \to Z_{R/T} \to Z_{R/k} \to Z_{T/k} \to 0$$

is exact (where $Z_{R/T} \hookrightarrow H^1(X, T^o)$).

$$Z_{R/K} \hookrightarrow Ex(X/k, \mathcal{O}_X), \; Z_{T/k} \hookrightarrow H^o(X, T^1)),$$

and R is a formal power series ring over T in $m = \dim_k H^1(X, T^o)$ variables. Thus R is regular (formally simple over k) if and only if T is regular. Thus we are reduced to investigating T, or more precisely, G.

The local functor G decomposes into the disjoint sum of "punctual" functors G_P for the singular point, $P \in X$. Specifically, let P be a singular point of x, and define $G_P : \underline{C} \to \underline{(Sets)}$ by the formula

$$G_P(A) = \underline{Def}\,(X/k, \text{Spec } A)_P \quad \text{for } A \in \underline{C}$$

that is, G_P is the stalk of $\underline{Def}$ at P. Then $G_P(D) = T^1_P$ has finite dimension over k, since P is an isolated singularity. Thus G_P is half exact (§3. 2, 3. 3), and has a projective hull (T_P, η_P).[1]

Now $\underline{Def}(X/k, \mathcal{O}_X)$ is a "sky-scraper" sheaf, concentrated at the finite number of singular points $P_1, \ldots, P_n$ of X. In other words, If U is an open set in X containing none of the P_i, then $H^o(U, \underline{Def}(X/k, A)) =$

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[1] G_P is also the formal moduli functor of Spec $\mathcal{O}_{P/k}$.

(e_U), the trivial section of $\underline{\mathrm{Def}}(U/k,\ A)$ (4. 0). Since $H^1(X,\ T^1) = (0)$, the map $h \rightsquigarrow (h_{P_i})_{1 \leq i \leq n}$ induces a bijection $H^0(X,\ \underline{\mathrm{Def}}(X/k,\ A)) \hookrightarrow \prod_{i=1}^{n} \underline{\mathrm{Def}}(X/k,\ A)_{P_i}$ that is

$$(1) \qquad G \hookrightarrow \prod_{i=1}^{n} G_{P_i}$$

It is clear that (1) extends to the "completed" functors (3. 2): $\hat{G} \hookrightarrow \prod_{i=1}^{n} \hat{G}_{P_i}$ on $\hat{\underline{C}}$. Let $(T_{P_i},\ \eta_{P_i})$ be a projective hull for G_{P_i}; let $T = T_{P_1} \hat{\otimes}_k \cdots \hat{\otimes}_k T_{P_n}$, let η_i be the image of η_{P_i} under the canonical morphism $T_{P_i} \to T$, and let $\eta = \eta_1 \times \dots \times \eta_n \in \hat{G}(T)$ via (1). Then by the universal mapping property* of complete tensor products, $(T,\ \eta)$ is a projective hull of G. Using the same universal mapping property again, we see that T is formally simple over k if and only if each T_{P_i} is.

Let $P \in X$ be a fixed singular point of X. Then by Thm. 1, iii, 2. 3, T_P will be formally simple over k if $T_P^2 = (0)$. Therefore, by the corollary to Proposition 1, 4. 1, (h 60) we have

Proposition 2. Let X be a reduced curve over k. Then T (and hence R) will be formally simple over k in the following cases:

(a) X is locally a complete intersection.

(b) X is locally of codimension 2; that is, each point of X has a neighborhood which may be imbedded in a nonsingular three fold.

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*If $A,\ B \in \underline{C}$, $A \times B$ is the direct sum of A and B in $\underline{C}$.

Example 1. If X is a plane affine curve with equation $f(X, Y) = 0$ and coordinate ring $B = k[X, Y]/(f(X, Y)) = k[x, y]$, then $T^1_{B/k} = \mathrm{Ext}^1_B(\Omega_{B/k}, B) \cong B/\left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}\right)$. Thus if $f(X, Y) = Y^2 - X^n$, and $\underline{p}$ is the origin, then the generic linear deformation (p.48) of X/k in the neighborhood of $\underline{p}$ has equation

$$y^2 = x^n + t_0 + t_1 x + \dots + t_{n-2}x^{n-2}$$

over $k[t_0, \dots, t_{n-2}]/\underline{t}^2$. (Assuming $p \neq 2$, $p \nmid n$, if p is the characteristic of k.)

Example 2. We give an example of a singular point $\underline{p}$ on a curve X such that $T_{\underline{p}}$ (and hence R) is not a regular local ring.

In general, if $A \to B$ is a flat ring homomorphism, and C is a deformation of B/A to $D = A[\mathcal{E}]/(\mathcal{E}^2)$, then the obstruction to deforming C further to $A[\mathcal{E}]/(\mathcal{E}^3)$ may be computed as a "Yoneda Product" $[\theta, \theta] \in T^2(B/A, B)$ where $\theta \in T^1(B/A, B)$ is the class of C. We give an explicit determination of this obstruction in terms of "generators and relations". Write

$$(1) \qquad 0 \to I \to P \to B \to 0$$

where $P = A[x]$ is a polynomial ring. Let (f_i) be generators of I and let F be a free P module on symbols e_i.

$$(2) \qquad 0 \to R \to F \to I \to 0$$

Similarly, for C/D we have

$$(3) \qquad 0 \to J \to Q \to C \to 0$$

where $Q = D[x]$.

Now J has generators $f_i + \mathcal{E}\Delta f_i$ where $\Delta f_i \in P$, and the Δf_i must satisfy the relation

$$\text{(4r)} \qquad \sum r_i(\Delta f_i) \in I$$

for each $r = \sum r_i e_i \in R$.

Thus there exist elements $\Delta r_i \in P$ such that

$$\text{(5r)} \qquad \sum r_i(\Delta f_i) + \sum(\Delta r_i)f_i = 0\,.$$

The class of $\Delta r = \sum \Delta r_i e_i$ in $F/R = I$ is independent of the choice of Δr_i in (5r). Thus the assignment $r \rightsquigarrow \sum \Delta r_i \Delta f_i \mod I$ is a homomorphism h of R into B. It is clear that h vanishes on the submodule R_0 of R generated by the elements $f_i e_j - f_j e_i$. Thus h induces a homomorphism $h_0 : R/R_0 \longrightarrow B$. The class of h_0 in $T^2(B/A, B)$ (see 1.1) is the desired obstruction.

If $A = k$, we call the obstruction obtained above a <u>primary</u> obstruction to deforming B (since the obstruction to deforming B to $k[\mathcal{E}]/(\mathcal{E}^2)$ is always zero.)

We now give an example of an integral affine curve[1] $X = \operatorname{Spec} B$ for which there exists a non vanishing primary obstruction. Let X be the parametric curve $x = t^7$, $y = t^8$, $z = t^9$, $w = t^{10}$ in $A^4 k$. In other words let B be the subring of $k[t]$ (polynomial ring) generated over k by

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[1] An example in dimension zero is obtained by taking $B = k[x_1, x_2, x_3]/(\underline{x})^2$. For the artin ring $B' = k[x, y]/(x^2, xy, y^2)$, $T^2_{B'} \neq (0)$, but $T(=R)$ is formally simple.

t^7, t^8, t^9, t^{10}. (The isolated singularity we have in mind is the origin $t = 0$). If we write B as a quotient P/I where $P = k[x, y, z, w]$, we find that I is generated by six elements $f_1 = y^2 - xz$, $f_2 = xw - yz$, $f_3 = z^2 - yw$, $f_4 = x^4 - w^2y$, $f_5 = x^3y - zw^2$ and $f_6 = w^3 - x^3z$. (It is clear that the f_i are in I. To see that they generate I, one could use a technique outlined by Northcott in [ ], p. ? .) Let K (= $k(t)$) be the quotient field of B. Let F be the free P module on $e_1, \dots, e_6$. Write

$$(6) \qquad 0 \to R \to F \to I \to 0 \qquad \text{and}$$

$$(\bar{6}) \qquad 0 \to \bar{R} \to \bar{F} \to I/I^2 \to 0$$

where $\bar{F} = F/IF$, and $\bar{R} = R/(IF \cap R)$. Since B is reduced, $T^2_B = \mathrm{Ext}^1_B(I/I^2, B)$ (4.1) and the obstructing homomorphism $h_0 : R/R_0 \to B$ is actually a homomorphism $\bar{h} : \bar{R} \to B$. We list only the generators of $\bar{R} \subset \bar{F}$. $\bar{R}$ is given by the matrix

$$(7) \qquad \begin{array}{c|c|c|c|c|c|c|}
 & e_1 & e_2 & e_3 & e_4 & e_5 & e_6 \\ \hline
a & t^9 & t^8 & t^7 & & & \\ \hline
b & t^{20} & & & t^8 & -t^7 & \\ \hline
c & & & t^{20} & & t^9 & t^8 \\ \hline
ta & t^{10} & t^9 & t^8 & & & \\ \hline
tb & t^{21} & & & t^9 & -t^8 & \\ \hline
tc & & & t^{21} & & t^{10} & t^9 \\ \hline
\end{array}$$

(Corresponding elements of R may be obtained by replacing the entries with monomials in x, y, z, w. For example, in the second row (b) replace t^{20} by w^2, etc.) From ($\overline{6}$) we see that Rank $\overline{R}$ (= $\mathrm{Dim}_k(\overline{R} \otimes K)$) = 3. The table (7) shows that a, b, c are linearly independent elements of $\overline{R}$. Hence $\mathrm{Hom}(\overline{R}, B)$ is a submodule of the free module B^3, and the elements of $\mathrm{Hom}(\overline{R}, B)$ will be denoted by three tuples $(\alpha, \beta, \gamma) \in B^3$ (representing the images of a, b and c.)

In this notation, we see that the image of $\mathrm{Hom}(\overline{F}, B)$ in $\mathrm{Hom}(\overline{R}, B)$ is generated by $(0, 0, t^8)$, $(0, -t^7, t^9)$, $(0, t^8, 0)$, $(t^7, 0, t^{20})$, $(t^8, 0, 0)$ and $(t^9, t^{20}, 0)$. We claim that there is a deformation C of B to $k[\mathcal{E}]/\mathcal{E}^2$ for which the associated primary obstruction is $(0, 0, -t^{20})$. Since $(0, 0, -t^{20})$ is not in the image of $\mathrm{Hom}(\overline{F}, B) \longrightarrow \mathrm{Hom}(\overline{R}, B)$ it determines a nonzero element of $\mathrm{Ext}^1(I/I^2, B)$, and we would be done.

Let $M = \mathrm{Image}(I/I^2 \longrightarrow \Omega_{P/k} \otimes_P B)$ be the submodule generated by the df_i's. Then $T^1_B = \mathrm{Ext}^1(\Omega_{B/k}, B)$ is the cokernel of $\mathrm{Hom}(\Omega_{P/k}, B) \longrightarrow \mathrm{Hom}(M, B)$. By computing the df_i's it may be shown that the system $\Delta f_1 = \Delta f_2 = \Delta f_3 = 0$, $\Delta f_4 = zw$, $\Delta f_5 = w^2$, $\Delta f_6 = -x^3$, when reduced mod I, determine a homomorphism $M \longrightarrow B$ ($df_i \rightsquigarrow \Delta f_i \pmod{I}$), and therefore, a deformation C of B/k to $k[\mathcal{E}]/(\mathcal{E}^2)$. If the procedure outlined above is followed, one finds that the homomorphism $\overline{h}$ is given by $(0, 0, -t^{20})$, and our example is thereby completed.

4.3. The dimension of R.

4.3.0. Let X be a reduced curve over k, and let R be the projective hull of its formal moduli functor. The formula $\dim_k Z_{R/k} = \dim_k H^0(X, T^1) + \dim_k H^1(X, T^0)$ (4.1) for the dimension of the Zariski tangent space of R is not very convenient, since neither of the two terms involved is easily computed, even for plane curves. In general, we cen get a better formula by the following device. Let $i : X \hookrightarrow Y$ be a closed immersion of X into a scheme Y simple and proper over k (for example, we could take $Y = \mathbb{P}^n_k$ for suitable n.) By the fundamental exact sequence for the T^i's (2.20) we have

$$(8) \qquad 0 \to T^o_X \to i^*T^o_Y \to T^1_{X/Y} \to T^1_X \to 0 \qquad \text{(exact)}$$

where
$$T^o_X = \underline{\mathrm{Hom}}_{\mathcal{O}_X}(\Omega_{X/k}, \mathcal{O}_X)$$
$$T^o_Y = \underline{\mathrm{Hom}}(\Omega_{Y/k}, \mathcal{O}_Y), \; i^*T^o_Y = \underline{\mathrm{Hom}}_{\mathcal{O}_X}(i^*\Omega_{Y/k}, \mathcal{O}_X)$$
$$T^1_{X/Y} = \underline{\mathrm{Hom}}_{\mathcal{O}_X}(I/I^2, \mathcal{O}_X) \quad (\mathcal{O}_X = \mathcal{O}_Y/I)$$

If χ denotes euler characteristic, we get

$$(9) \qquad \chi(T^1_X) - \chi(T^o_X) = \chi(i^*T^o_Y) - \chi(T^1_{X/Y}) \qquad \text{or}$$

$$(10) \qquad \dim_k Z_{R/k} - \dim_k H^o(X, T^o_X) = \chi(i^*T^o_Y) - \chi(T^1_{X/Y}).$$

Now $\dim_k H^o(X, T^o_X)$ is less than or equal to three times the number of irreducible components of X (at least in characteristic zero) and is zero if the genus of the normalization of each irreducible component of X is > 1 (in characteristic 0.) If X is locally a complete intersection (whence $\dim_k Z_{R/k} = \dim R$) the right hand side of (10) is $3(p_a - 3)$, where p_a is the arithmetic genus of X (4.3).

In order to estimate the right hand side of (10), we prove a Riemann-Roch theorem for torsion free sheaves on a reduced curve, by an óbvious extension of the methods of Oort in [8].

4.3.1. Genus formula.

Let X be a reduced curve, and let X_i, $i = 1, 2, \ldots, n$ be the irreducible components of X.

Let $q_i : X_i \hookrightarrow X$ be the closed immersion of X_i into X, and let $\mathcal{K}_i$ be the constant quotient field sheaf on X_i. Then $\mathcal{K} = \sum \oplus q_{i*}\mathcal{K}_i$ is the sheaf of total quotient rings of $\mathcal{O}_X$; that is, for each affine open $U = \operatorname{Spec} B \subset X$, the total quotient ring of B is $\Gamma(U, \mathcal{K})$, and for each $x \in X$, $\mathcal{K}_x$ is the total quotient ring of $\mathcal{O}_X$.

Let $\tilde{X}_i$ be the normalization of X_i, $\tilde{X} = \coprod_i \tilde{X}_i$ (disjoint sum) and $\pi : \tilde{X} \to X$ the projection. Then $\mathcal{O}_{\tilde{X}} = \sum\oplus \mathcal{O}_{\tilde{X}_i}$, $\pi_*\sum\oplus\mathcal{K}_i = \mathcal{K}$, and $\pi_*\mathcal{O}_{\tilde{X}}$ is the integral closure of $\mathcal{O}_X$ in $\mathcal{K}$ (meaning that for each affine open $U = \operatorname{Spec} B \subset X$, $\tilde{B} = \Gamma(U, \pi_*\tilde{\mathcal{O}}_X)$ is the integral closure of B in its total ring of quotients $\Gamma(U, \mathcal{K})$). The sheaf $\pi_*\tilde{\mathcal{O}}_X/\mathcal{O}_X$ is a torsion $\mathcal{O}_X$ module, zero except at the singular points of X. We define

$$p_a = p_a(X) = 1 - \chi(X, \mathcal{O}_X) \quad \text{(arithmetic genus)}$$

$$g = p_a(\tilde{X}) = \sum g_i - n + 1 \quad \text{where } g_i = \text{genus of } \tilde{X}_i \tag{11}$$

$$\delta = H^0(X, \pi_*\tilde{\mathcal{O}}_X/\mathcal{O}_X) = \sum_{x\in X} \ell\left(\pi_*\mathcal{O}_{\tilde{X}}/\mathcal{O}_X\right)_x$$

Since π is a finite (in particular, affine) morphism, $H^i(\tilde{X}, \mathcal{O}_{\tilde{X}}) = H^i(X, \pi_*\mathcal{O}_{\tilde{X}})$; thus the exact sequence

$$0 \to \mathcal{O}_X \to \pi_*\mathcal{O}_{\tilde{X}} \to \pi_*\mathcal{O}_{\tilde{X}}/\mathcal{O}_X \to 0$$

shows that

(12) $$p_a = g + \delta \qquad \text{(genus formula)}$$

4.3.2. Riemann Roch Theorem.

We maintain the notation of the preceding section. A coherent $\mathcal{O}_X$ module $\mathcal{F}$ is said to be torsion free if $\mathcal{F}(U)$ is a torsion free $\mathcal{O}_X(U)$ module (4.1) for each affine open U. In other words $\mathcal{F}$ is torsion free if and only if the map $\mathcal{F} \to \mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{K}$ is injective. We say that a torsion free sheaf $\mathcal{F}$ has rank r if $\mathcal{F}_x$ has rank r at each generic point x of X. Thus if $\mathcal{F}$ has rank r, $\mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{K} \cong \mathcal{K}^r$, and $\mathcal{F}$ may be regarded* as a sub $\mathcal{O}_X$ module of $\mathcal{K}^r$. Note that in this case $\mathcal{F}/\mathcal{F} \cap \mathcal{O}_X^r$ and $\mathcal{O}_X^r/\mathcal{F} \cap \mathcal{O}_X^r$ are torsion modules, with support at a finite number of closed points.

Definition. Let $\mathcal{F}$ be a torsion free sheaf of rank r, and choose an imbedding $\mathcal{F} \subset \mathcal{K}^r$. We define the degree of $\mathcal{F}$ by the formula

(12) $$\deg(F) = H^0(X, \mathcal{F}/\mathcal{F} \cap \mathcal{O}_X^r) - H^0(X, \mathcal{O}_X^r/\mathcal{F} \cap \mathcal{O}_X^r).$$

From the exact sequences

$$0 \to \mathcal{F} \cap \mathcal{O}_X^r \to \mathcal{F} \to \mathcal{F}/\mathcal{F} \cap \mathcal{O}_X^r \to 0 \qquad \text{and}$$

$$0 \to \mathcal{F} \cap \mathcal{O}_X^r \to \mathcal{O}_X^r \to \mathcal{O}_X^r/\mathcal{F} \cap \mathcal{O}_X^r \to 0$$

It follows that

(13) $$\chi(\mathcal{F}) = \chi(\mathcal{O}_X^r) + \deg(\mathcal{F})$$

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*By choosing r linearly independent elements of $\mathcal{F}_x$ over $\mathcal{K}_x$ at each generic point x.

Thus deg ($\mathcal{F}$) depends only on $\mathcal{F}$, and not on the imbedding $\mathcal{F} \subset \mathcal{K}^r$ chosen. Recalling the definition of p_a (4.31) we see that (13) may be written

(R.R) $$\chi(\mathcal{F}) = \mathrm{rank}\,(\mathcal{F})\,(1 - p_a) + \deg\,(\mathcal{F})$$

Properly speaking, this is only the Riemann part of a Riemann-Roch theorem. To get the Roch (duality) part we appeal to Grothendieck's duality theorem, which we state in the following form:

Duality Theorem [4]. Let X be a reduced Cohen Macauley scheme of dimension m, and let $i : X \hookrightarrow Y$ be a closed immersion of X into a non singular projective variety Y of dimension n.

Put $w_X = \underline{\mathrm{Ext}}^q_{\mathcal{O}_Y}(\mathcal{O}_X, \Lambda^n \Omega_Y)$, where q is the codimension of X in Y. Then for any coherent sheaf $\mathcal{F}$ on X there are perfect pairings

$$H^k(X, \mathcal{F}) \times \mathrm{Ext}^{m-k}_{\mathcal{O}_X}(\mathcal{F}, w_X) \to k$$

$0 \leq k \leq m = \dim X.$

Further remarks. w_X does not depend on the immersion $i : X \hookrightarrow Y$ of X into a nonsingular variety. By [5], w_X is a torsion free $\mathcal{O}_X$ module. If $\mathcal{O}_X = \mathcal{O}_Y/I$, there is a canonical homomorphism

(14) $$\underline{\mathrm{Hom}}_X(\Lambda^q I/I^2, i^* \Lambda^n \Omega_Y) \to w_X$$

which is an isomorphism if X is locally a complete intersection. This may be seen, locally, by comparing a projective $\mathcal{O}_Y$ resolution of $\mathcal{O}_X$ with a Koszul complex resolution of $\mathcal{O}_X$. If X is a reduced curve, Grothendieck has shown that w_X is the sheaf of Rosenlicht differentials

([9],) on X.

Returning to our situation, where X is a reduced curve, we see by (R. R) that

$$\deg(w_X) = 2p_a - 2. \tag{15}$$

In order for the formula (R. R) to be of any use, we must establish a few properties of deg $\mathcal{F}$. X is a reduced curve in all of what follows.

Lemma 1. Let $\mathcal{F}$ be a torsion free sheaf of rank r on X, and $\mathcal{L}$ an invertible sheaf on X. Then

$$\deg(\mathcal{L} \otimes \mathcal{F}) = r.\deg \mathcal{L} + \deg \mathcal{F}$$

Proof. Pick imbeddings $\mathcal{L} \subset \mathcal{K}$ and $\mathcal{F} \subset \mathcal{K}^r$. It suffices to consider the contribution $\deg_x$ to the degree at a single closed point $x \in X$.

There exist nonzero divisors a, b in $\mathcal{O}_x$ such that $\mathcal{L}_x = a/b\, \mathcal{O}_x$. Hence $\deg_x(\mathcal{L}) = \ell(\mathcal{O}_x/b) - \ell(\mathcal{O}_x/a)$, $(\mathcal{L} \otimes \mathcal{F})_x = a/b\, \mathcal{F}_x$, so we may assume b = 1. Put $\mathcal{O}_x = A$, $\mathcal{F}_x = M$. From

$$\deg_x(\mathcal{L} \otimes \mathcal{F}) = \ell(aM/aM \cap A^r) - \ell(A^r/aM \cap A^r),$$

$$\deg_x(\mathcal{F}) = \ell(M/M \cap A^r) - \ell(A^r/M \cap A^r) \qquad \text{and}$$

$$0 \to M \cap A^r/(aM \cap A^r) \to A^r/(aM \cap A^r) \to A^r/(M \cap A^r) \to 0$$

$$0 \to M \cap A^r/(aM \cap A^r) \to M/(aM \cap A^r) \to M/(M \cap A^r) \to 0$$

It follows that

$$\deg_x(\mathcal{L} \otimes \mathcal{F}) - \deg_x(\mathcal{F}) = -\ell(M/aM) \tag{16}$$

We claim that $\ell(M/aM) = r \cdot \ell(A/aA)$. Suppose first that $M \subset A^r$. Then

$$0 \to aA^r/aM \to A^r/aM \to A^r/aA^r \to 0 \qquad \text{and}$$

$$0 \to M/aM \to A^r/aM \to A^r/M \to 0$$

show that $\ell(M/aM) = \ell(A^r/aA^r) = r \cdot \ell(A/aA)$. ($aA^r/aM \cong A^r/M$, since M is torsion free.) Now for arbitrary M, there exists a nonzero divisor $c \in A$ such that $cM \subset A^r$. But $M/aM \cong cM/acM$, and we are done by (16).

Corollary. If $\mathcal{L}$ is an invertible sheaf, and $\check{\mathcal{L}} = \underline{\mathrm{Hom}}(\mathcal{L}, \mathcal{O}_X)$, then $\deg \check{\mathcal{L}} = -\deg \mathcal{L}$.

Proof. $\mathcal{L} \otimes \check{\mathcal{L}} \cong \mathcal{O}_X$.

Remark. The corollary is false, in general, if $\mathcal{L}$ is only a torsion free sheaf of rank 1. In fact, the statement "$\deg \check{\mathcal{F}} = -\deg \mathcal{F}$ for any torsion free sheaf $\mathcal{F}$ of rank 1 on X" characterizes Gorenstein curves.[1]

We will need the following for the proof of proposition 1 below.

Lemma 2. Let A be a reduced noetherian ring. Let $x \in A$, $x \neq 0$ and let I be an ideal in A such that (x, I) contains a non-zero divisor. Then there exists $y \in I$ such that $x + y$ is a non-zero divisor.

Proof. Let $\mathfrak{q}_1, \ldots, \mathfrak{q}_n$ be the primes associated to (0). Say $x \in \mathfrak{q}_i$ $1 \leq i \leq r$ (we may assume x is a zero divisor) and $x \notin \mathfrak{q}_j$, $j > r$. Choose $z \in \bigcap_{j>r} \mathfrak{q}_j$ such that $z \notin \bigcup_{1 \leq i \leq r} \mathfrak{q}_i$. Since $I \not\subset \mathfrak{q}_i$ $1 \leq i \leq r$ (otherwise every element of (x, I) would be a zero divisor), there exists $w \in I$, $w \notin \bigcup_{1 \leq i \leq r} \mathfrak{q}_i$. Then $y = zw$ will do, since $x + y \notin \bigcup_{1 \leq i \leq r} \mathfrak{q}_i$.

Proposition 1. Let $\mathcal{F}$ be a locally free sheaf of rank r on a reduced curve X. Then

$$\deg \mathcal{F} = \deg \Lambda^r \mathcal{F}$$

Proof. As in Lemma 1, the question is "local". Let $x \in X$, $A = \mathcal{O}_x$,

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[1] 2

$M = \mathcal{F}_x$. After multiplying M by a suitable non-zero divisor, we may assume by Lemma 1, that $M \subset A^r$. We have to show that

$$\ell(A^r/M) = \ell(A/\det M). \tag{17}$$

The proof is by induction on $r = \operatorname{rank} M$. For $r = 1$, (17) is trivially true; assume that (17) holds for all modules of rank < rank M.

M has a basis $x_1, \ldots, x_r$, where $x_i = \sum a_{ij} e_j$ ($e_j = j$th ~~is a~~ unit vector in A^r), and $\det M = a = \det(a_{ij})$. If (a_{ij}) has the form

$$(a_{ij}) = \begin{pmatrix} a_{11} & \\ 0 & (b) \end{pmatrix} \tag{18}$$

then we may let M' be the submodule of A^{r-1} determined by (b), and $M'' = a_{11}A$. Then

$$0 \rightarrow M' \rightarrow M \rightarrow M'' \rightarrow 0 \qquad \text{induces}$$

$$0 \rightarrow A^{r-1}/M' \rightarrow A^r/M \rightarrow A/M'' \rightarrow 0 .$$

and (17) follows by the induction hypothesis.

If (a_{ij}) does not have the form (18), we can reduce to that case as follows. The first column of (a_{ij}) has a non-zero entry, say a_{11}. Let $I = (a_{21}, \ldots, a_{r1})$ be the ideal generated by the rest of the first column entries. Then (a_{11}, I) contains a non-zero divisor, namely $a = \det M$. Thus by Lemma 2, there exist c_i $(2 \leq i \leq r)$ in A such that $a_{11} + \sum_{i=2}^{r} c_i a_{ij}$ is a non-zero divisor. Thus by changing the basis of M, we may assume that a_{11} is not a zero divisor. Suppose $a_{12} \neq 0$. Then, if we let M' be the submodule of M generated by $x_1, a_{11}x_2, x_3, \ldots, x_r$

we find that M' has a matrix (b) with $b_{11} = a_{11}$, $b_{12} = 0$, and that $M/M' \cong A/a_{11}$. $0 \rightarrow M/M' \rightarrow A^r/M' \rightarrow A^r/M \rightarrow 0$ shows that if (17) holds for M', then it holds for M. Thus we may reduce to the case where all first column entries are zero except a_{11}, in which case (17) is all ready proven.

Now let X be a reduced curve, and R the projective hull of its formal moduli functor. Put $d = \dim_k Z_{R/k}$, and $h^o = \dim_k H^o(X, T^o)$. Let $i : X \hookrightarrow Y$ be a closed immersion of X into a non-singular variety of dimension n, and put $w = \underline{\mathrm{Ext}}^{n-1}_{\mathcal{O}_Y}(\mathcal{O}_X, \Lambda^n \Omega_Y)$. In the notation of 4. 3. 1 (10) we have

$$d - h^o = \chi(T^1_{X/Y}) - \chi(i^*T^o_Y) \qquad (19)$$

Now $T^1_{X/Y} = \underline{\mathrm{Hom}}(I/I^2, \mathcal{O}_X)$ (resp. $i^*T^o_Y = \underline{\mathrm{Hom}}(i^*\Omega_Y, \mathcal{O}_X)$) is a torsion free (resp. locally free) sheaf of rank $n-1$ (resp. rank n). Hence by (R.R) and Proposition 1, we have

$$\begin{aligned} d - h^o &= p_a - 1 + \deg T^1_{X/Y} - \deg(i^*T^o_Y) \\ &= p_a - 1 + \deg T^1_{X/Y} + \deg(i^*\Lambda^n\Omega_Y) . \end{aligned}$$

Now if X is locally a complete intersection, then I/I^2 is locally free, and the isomorphism (14) shows that

$$d - h^o = p_a - 1 + \deg w = 3p_a - 3 .$$

Thus

<u>Theorem.</u> Let X be a reduced curve, <u>locally</u> a <u>complete</u> <u>intersection.</u> Then R is a power series ring in

$$d = 3p_a - 3 + h^o . \qquad (20)$$

variables, where $h^o = \dim_k H^o(X, T^o)$.

Example. The equality (20) does not hold in general, even if R is formally simple. Let X be the n-fold normal crossing in $\mathbb{P}^n$: n linearly independent lines $L_1, \ldots, L_n$ meeting in a point 0. The (affine) equations of X are

$$x_i x_j = 0 \qquad 1 \leq i < j \leq n$$

Let A be the local ring at the origin. If A_i is the local ring of 0 on L_i, then the integral closure $\tilde{A}$ of A is $\sum \oplus A_i$. Hence $\delta_0 = \ell(\tilde{A}/A) = n-1$. Therefore by the genus formula (12) of 4.31, $\underline{p_a = 0}$.

For each i, X has two global derivations given by $Dx_1 = x_i$ (resp. x_i^2), and $Dx_j = 0$, $j \neq i$. Thus $h^o = 2n$, and

$$3p_a - 3 + h^o = 2n - 3.$$

If we imbed T^o in $\mathcal{K}$ by choosing the generic derivations $D_i : D_i(x_i) = 1$ on L_i, (and extending to X by $D_i(x_j) = 0$ $i \neq j$) then T^o_A is generated by $E_1, \ldots, E_n$, with $E_i(x_j) = \delta_{ij} x_i$ (δ_{ij} = Kronecker delta), that is $E_i = x_i D_i$; thus the degree of T^o at 0 is $-\ell(A/\underline{m}A) = -1$. On the other hand, each point at infinity contributes +2 to det T^o, so

$$\deg T^o = 2n - 1$$

Therefore since $\dim H^o(X, T^o) = 2n$, we find by (R. R) that

$$\dim_k H^1(X, T^o) = 0$$

that is, all non-trivial deformations of X arise from the singularity at 0, and

$$d = \dim_k H^0(X, T^1) = \ell(T^1_A).$$

Now T^1_A = cokernel (Hom $(\Omega_P, A) \to$ Hom (I, A)) where $P = k[x_1, \ldots, x_n]$ (polynomial ring) and I is generated by $\alpha_{ij} = x_i x_j$ $i \neq j$ It is clear that the only relations on the α_{ij} are

$$x_i\alpha_{jk} + x_j\alpha_{ik}$$

Thus if $n \geq 3$, Hom (I, A) is generated over A by the homomorphisms $\langle \alpha_{ij} | j \rangle$ and $\langle \alpha_{ij} | i \rangle$, where $\langle \alpha_{ij} | j \rangle (\alpha_{ij}) = x_j$, $\langle \alpha_{ij} | i \rangle (\alpha_{ij}) = x_i$, and $\langle \alpha_{ij} | j \rangle (\alpha_{pq}) = \langle \alpha_{ij} | i \rangle (\alpha_{pq}) = 0$ for $\alpha_{ij} \neq \alpha_{pq}$. There are $2 \cdot \frac{(n)(n-1)}{2} = n \cdot (n-1)$ such symbols. Note that $x_k \langle \alpha_{ij} | j \rangle = x_k \langle \alpha_{ij} | i \rangle = 0$ if $k \neq i, j$.

The image of Hom (Ω_P, A) is generated by $\delta_1, \ldots, \delta_n$, where $\delta_i = \sum_{j \neq i} \langle \alpha_{ij} | j \rangle$ clearly $x_k \delta_i = x_k \langle \alpha_{ik} | k \rangle$, so that $\underline{m}\, T^1_A = (0)$. The δ_i are obviously linearly independent over k. Thus $\ell(T^1_A) = n \cdot (n-1) - n = n \cdot (n-2)$ if $n \geq 3$, and

$$d = \dim_k H^0(X, T^1) + \dim_k H^1(X, T^0) = n \cdot (n-2)$$

so that

$$d - (3p_a - 3 + h^0) = \begin{cases} (n-3)(n-1) & \text{if } n \geq 3. \\ = 0 \text{ if } n = 2, 3. & \end{cases}$$

A straightforward calculation also shows that $T^2 = \operatorname{Ext}^1(I/I^2, A) = (0)$ for $n \geq 2$; hence R is a power series ring in d variables over k.

Remark 1. The ring R does not pro-represent the X above. Consider the case $n = 2$ for example. $(xy = 0)$. Since $H^1(X, T^0) = (0)$, all the obstructions to lifting automorphisms (§3.4) are local; i.e. they lie in $T^1_A \otimes J$. Consider the (global) automorphism of $xy + \mathcal{E} = 0$ over $k[\mathcal{E}]/\mathcal{E}^2$ given by $x \rightsquigarrow x + x\mathcal{E}$, $y \rightsquigarrow y$. This does not extend to an automorphism of $xy + \mathcal{E} = 0$ over $k[\mathcal{E}]/\mathcal{E}^3$, even in the neighborhood of 0. For if it did, there would exist $a, b \in A$ such that $x \rightsquigarrow x + x\mathcal{E} + a\mathcal{E}^2$, $y \rightsquigarrow y + b\mathcal{E}^2$ defines an automorphism of $xy + \mathcal{E} = 0$ over $k[\mathcal{E}]/\mathcal{E}^3$; by substitution we would get $ay + bx = -1$ in A, which is impossible.

<u>Remark 2.</u> The comparison of $d = \dim Z_{R/k}$ and $(3p_a = 3 + h^0)$ may be determined locally. Namely, it is a question of comparing the degrees of w and $\underline{\mathrm{Hom}}\,(I/I^2, i^*\Lambda^n\Omega_Y)$. This may be done by choosing an inclusion $w \subset \mathcal{K}$, which induces $\underline{\mathrm{Hom}}\,(\Lambda^{n-1}I/I^2, i^*\Lambda^n\Omega_Y) \subset w \subset \mathcal{K}$. (via c)), which in turn induces $\underline{\mathrm{Hom}}\,(I/I^2, i^*\Lambda^n\Omega_Y) \subset \mathcal{K}^{n-1}$. Then the degrees of the two sheaves may be compared at any point of X.

It is conjectured that $\rho = d - (3p_a - 3 + h^0) \geq 0$ for any reduced curve. I have no candidate for a "$\rho = 0$" condition. For the parametric (Gorenstein) curve $x = t^5$, $y = t^6$, $z = t^7$, $w = t^8$, $\rho = 7$; (and $T^2 = (0)$); for the non-Gorenstein curve $x = t^3$, $y = t^4$, $z = t^5$ (resp. $x = t^3$, $y = t^5$, $z = t^7$) $\rho = 0$ (resp. $\rho = 1$).

§4.3. Pro-representability of the formal moduli functor.

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The numerical (sufficient) condition $H^0(X, T^0) = (0)$ (i.e., $S = R$ in the notation of 3.4) for pro-representing the formal moduli functor of $X|k$ is rather restrictive. However, it gives us some partial results for reduced curves in characteristic zero, due to the following theorem of Seidenberg.

Theorem (Seidenberg [ ]) Let A be a reduced noetherian ring <u>containing</u>

the rational numbers, and let $\tilde{A}$ be the integral closure of A in its total ring of quotients. Then every derivation $D : A \rightarrow A$ extends to $\tilde{D} : \tilde{A} \rightarrow \tilde{A}$.

Corollary. Let X be a reduced curve over k, where chc. $k = 0$, let $\tilde{X} = \coprod \tilde{X}_i$ be the normalization of X, and let $\pi : \tilde{X} \rightarrow X$ be the projection. Then there is an inclusion $T^o_X \hookrightarrow p^* T^o_{\tilde{X}} = p_* \sum_i \oplus T_{\tilde{X}_i}$ which induces $H^o(X, T^o_X) \subseteq \sum_i H^o(X, T^o_{\tilde{X}_i})$.

Now for a non-singular, irreducible curve X of genus g, $\dim H^o(X, T^o_X) = 3$ if $g = 0$, $= 1$ if $g = 1$, $= 0$ if $g > 1$. Thus

Theorem. Let X be a reduced curve. Then the formal moduli functor of $X | k$ is pro-representable in the following cases.

(a) The characteristic of k is 0, and the genus of each irreducible component of the normalization of X is > 1.

(b) X is non-singular $(\tilde{X} = X)$.

The only case not accounted for by the previous remarks is that of an irreducible non-singular curve of genus one. For this we may appeal to the much more general result of Mumford [7]:

Theorem. Let $X | k$ be an abelian variety of dimension g. Then the formal moduli functor F of $X | k$ is pro-represented by a regular local ring of dimension g^2. (The functor F is the same, whether we take deformations of $X | k$ qua abelian variety or not.)

Example. Let X be the plane curve

$$y^2 = g(x^p) \qquad p \neq 2$$

(in affine coordinates), where $g(t)$ is a separable polynomial of degree m.

Then the genus of $\tilde{X}$ is $\left[\frac{m-1}{2}\right]$, while $h^0 = 3$. An analysis similar to the one under Remark 1, 4.2 shows that the formal moduli functor is not representable.

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