Correspondence Atiyah ↔ Hirzebruch about *K*-theory

Abstract

Transcribed by Shu Otsuka

1. Letter Atiyah → Hirzebruch dated December 31 1957.

Trinity College
31st Dec. 1957

Dear Fritz,

Greetings to you for the new year! May 1958 see you & all the family flourishing.

Thank you for your Christmas card. The manuscript, long delayed is now almost ready. It has been typed and only a few correction remains. I hope to despatch it to you within a week.

Regarding next summer, I shall be very pleased to join in the "Bonn Colloquium" again. I cannot leave Cambridge till about *June 20th* since I have examinations to see to. However after that I am free, and could come at any suitable time to synchronize with other people. Will you get Bott this time? I believe he will definitely be coming to Edinburgh & so should be available.

I have within the last week or so proved a theorem which may interest you. I have not yet written it up but I think all the details are all right. The result is the following:

Let X be an algebraic surface in projective 3-space P^3 with only ordinary double points as singularities. (i.e., at a singularity the eqn. of X becomes $f_2(x, y, z)$ + higher terms = 0 where $f_2(x, y, z)$ is a non-singular quadratic form).

Resolve each singularity in the obvious way (e.g, by blowing up the point in P^3) and we get a non-singular surface X. The X is homeomorphic to any non-singular algebraic surface in P^3 of degree = degree X. In particular the Kummer surface (or Andreotti surface) is homeomorphic to the general quartic surface.

As far as I know this result is new. It is very remarkable because it fails in all dimensions $\neq 2$ as one easily verifies. It works for dimension 2 essentially because the 2-dimensional quadric is the only quadric which is fibered.

How is Mathematics with you in Bonn? Do you hear anything new? How about the famous paper with Borel?

Do you plan to bring the family to Edinburgh? We hope to see you all then even if we don't all come to Bonn.

Yours sincerely
Michael

2. Letter Atiyah → Hirzebruch dated September 8 1958.

Grilerell [? undecipherable]
8th September 1958

Dear Fritz,

A further request for a recommendation! You were so successful last time that I have come again. This time it is for the Institute (Princeton). I am applying for membership there for the fall term 1959, and I need 3 recommendations for "character & academic ability". Would you consider writing one on my behalf? If so it should be sent *direct to Miss Underwood* at Princeton.

I have a small problem for you: prove that $n > 2k$ the coefficients of x^{n-1} in $(1-e^{-x})^{n-r-1}$ (for $1 \leq r \leq k-1$) are all integers – $n \equiv 0 \bmod q_k$, where q_k is some integer depending only on k. Moreover find a nice formula for q_k. For low values of k I have checked that to find $q_2 = 2$, $q_3 = q_4 = 4!$, $q_5 = q_6 = 4!5!$. This looks like a problem which suits you. It has applications to problems of vector fields on spheres.

Regards to Inge & all the family.

Yours
Michael

3. Letter Atiyah → Hirzebruch dated September 29 1958.

19 Beaumont Road,
Cambridge.

29th September 1958.

Dear Fritz,

I have recently made some progress with developing a "Grothendieck Theory" for almost complex manifolds, and I thought you might be interested in the results. So far I have only got incomplete and preliminary results, and the final theory has yet to be properly developed, but the main results are :

- A very simple direct proof that the Todd genus is an integer,
- A definition of $f_!(1)$ for any almost complex map $f : Y \to X$,
  - A Grothendieck R-R theorem for $f_!(1)$,
  - A weak form of R-R Theorem in the form suggested by you (i.e. in terms of the R-R subgroup of the cohomology group).

Briefly the details are as follows.

Conventions and terminology. All spaces are supposed to be of a type satisfying the classification theorem, so that $B_{U(n)}$ will always be understood to mean $U(N+n)/U(N)\times U(n)$ for some large N, and similarly for $MU(n)$. All maps, except where stated to have base points, and if X is any space we denote by X^+ the union of X with a disjoint (base) point. Let $K = \mathbb{Z}\times B_U$ ($\mathbb{Z}$ denoting the integers), and take a base point in the component $0\times B_U$. Let $K(X^+)$ denote the group of homotopy classes of maps $X^+ \to K$. As usual S and Ω will denote suspension and loop space respectively. Then by Bott we have $\Omega^2(K) = K$.

(i) *Integrality of the Todd genus*.

Let ξ_k denote the Universal bundle on $B_{U(k)}$, and consider

$$\lambda_{-1}(\xi_k^*) = \textstyle\sum_i (-1)^i \lambda^i(\xi_k^*) \in K(B_{U(k)}).$$

Restricted to $B_{U(k\text{-}1)}$ $\xi_k = \xi_{k\text{-}1}\oplus 1$ and so $\lambda_{-1}(\xi_k^*) = \lambda_{-1}(\xi_{k-1}^*)\lambda_{-1}(1) = 0$ since $\lambda_{-1}(1) = 0$. Hence $\lambda_{-1}(\xi_k^*)$ defines an element of $K(MU(k))$, i.e. a map $P_k : MU(k)^* \to K$. Consider the induced homomorphism of homotopy : [Note: $MU(k)$ has a canonical base point]

$$P_{k*} : \pi_{2n}(MU(k)^*) \to \pi_{2n}(K) \cong \mathbb{Z}.$$

I assert that this is just the Todd genus associated to a given Thom class. In fact if $X \in S^{2n}$ is a representative manifold for the Thom class $\alpha \in \pi_{2n}(MU(k)^*)$, and if $\eta \in \pi_{2n}(K)$ is the element induced by α from $\lambda_{-1}(\xi_k^*)$, then we have the Grothendieck formula:

$$\mathrm{ch}(\eta) = \varphi^* T(X), \quad \varphi^* : H^*(X) \to H^*(S^{2n}) \text{ the Gysin homomorphism.}$$

3. Letter Atiyah → Hirzebruch dated September 29 1958 (continued).

On the other hand, by Bott,

$$\mathrm{ch}(\eta) = (1/n!)s_n(\eta) = P_k^*(\alpha)g,$$

where g Î $H^{2n}(S^{2n})$ is the generator corresponding to a definite choice of orientation.

Note that, just as in Milnor's proof, the manifold X has only to be generalized almost complex.

(ii) *Definition of* $f_!(1)$

Let X,Y be differentiable manifolds, Y almost complex and let $f : Y \to X$ be an almost complex map (no base point), i.e. the graph Γ_f Ì $X \times Y$ has a given almost complex structure in its normal bundle. Embed Y E^{2n}, (as in Milnor's construction we want tangent γ Å normal = I_{2n} as complex vector bundles) and consider Γ_f Ì $X \times E^{2n}$ with its normal bundle having the almost complex structure given by addition. This then defines a map (no base point) of $X \times E^{2n} \to MU(n+k)$ with $X \times \dot{E}^{2n} \to$ base point, and hence defines a map (with base point):

$$S^{2n}(X^+) \to MU(n+k) \quad (2k = \dim X - \dim Y).$$

By composition with P_{n+k} this gives a map $S^{2n}(X^+) \to K$, and so a map $X^+ \to \Omega^{2n}(K) = K$. By definition this element of $K(X^+)$ is $f_!(1)$.

One must of course prove that this is independent of (a) the embedding of Y in E^{2n}, and (b) of the integer n (assuming this is sufficiently large). The proof of (a) is as in Thom Theory, but the proof of (b) is non-trivial and expresses a significant relation between the Bott periodicity for B_U, the stability (suspension) property of the Thom complexes $MU(k)$, and the maps P_k. Essentially one has to verify the following. Let α_k and β_k be the maps $MU(k) \to \Omega^{2n}(K)$, $S^2(MU(k)) \to K$ given respectively by the compositions :

$$MU(k) \xrightarrow{P_k} K = \Omega^{2n}(K),$$

$$S^2(MU(k)) \xrightarrow{\text{inclusion}} MU(k+1) \xrightarrow{P_{n+1}} (K).$$

Then we must verify that these maps are adjoints of one another. This is easily done by computing Chern classes.

(iii) *R-R for* $f_!(1)$

One breaks up the proof, as in the case of Grothendieck, to (a) an inclusion Y Ì X, and (b) a suspension argument (which is the analogue of the projection $X \times P \to X$ of Grothendieck). For the inclusion the proof is essentially the same as the proof of the integrality of the Todd genus given above; one has simply to use the Grothendieck formula:

$$\mathrm{ch}(\lambda_{-1}\xi_k^*) = c_k(\xi_k)T(\xi_k)^{-1}.$$

3. Letter Atiyah → Hirzebruch dated September 29 1958 (continued).

For the suspension we proceed as follows. We have $Y = \Gamma_f$ Î $S^{2n}(X^+)$, and so a map $1_Y \rightarrow i_!(1_Y)$ Î $K(S^{2n}(X^+))$ - even though $S^{2n}(X^+)$ is not a manifold this is clearly defined. Then one has simply to check in the diagram :

$$\begin{array}{ccc} K(X^+) & \xrightarrow{\text{ch}} & H(X) \\ S^{2n}\downarrow & & \downarrow S^{2n} \\ K(S^{2n}(X^+)) & \xrightarrow{\text{ch}} & H(S^{2n}(X^+)), \end{array}$$

(which is commutative as one sees from Bott) that $f_!(1) \xrightarrow{S^{2n}} i_!(1)$.

Of course, taking X = point, we see that the Todd genus of Y is just the dimension or augmentation of $f_!(1)$, and so is necessarily an integer.

(iv) *General R-R*.

Let X be fixed and consider all $f_!(1_Y)$ for variable Y and f. A singular cycle on X will be a formal sum of such maps. Then one can prove the following: the homomorphism of singular cycles → K(X) is an *epimorphism* (cf. the similar but different result in Algebraic Geometry). The proof is essentially the G/T argument of Hirzebruch-Borel. From this it follows formally that given $f : Y \rightarrow X$ and given y Î $K(Y^+)$ there is an element x Î $K(X^+)$ such that the R-R Theorem holds: $f_*(\text{ch}(y)T(f)) = \text{ch}(x)$. If X, Y are both almost complex and f is compatible with their almost complex structures then this is equivalent to the usual form of R-R. The unsatisfactory feature of this is that I know very little about $y \rightarrow x$; in particular is it really functorial (i.e. transitive)?

There are many points on which I am still not clear. In particular I would like to understand the exact nature of the Bott isomorphism $K \rightarrow \ \Omega^{2k}$ in this context. I hope also that there will be interesting applications to integrality questions, but on this you will know more than me.

Any comments or suggestions would be most welcome.

Yours sincerely,
Michael.

P.S. On re-reading this I find there is some confusion about base points – I think a little goodwill is needed here.

4. Letter Hirzebruch → Atiyah dated October 7 1958.

7. Oktober 1958

Hirzebruch to Atiyah

Excerpt
(slightly revised)

Your proof of the integrality of the Todd genus is extremely elegant. Thus this is reduced to Bott's divisibility theorem. In the joint paper with Borel the situation is exactly opposite. We start with the integrality of Todd (i.e. practically with the integrality of the index) and arrive at Bott's theorem exc [? undecipherable] 2 (Milnor's Todd gives then the complete Bott)...
Your method can be used to prove that the $\hat{A}$-genus is an integer if the second Stiefel-Whitney class vanishes and to show that $\hat{A}(X)$ *is an even integer if* $w_2 = 0$ *and* $\dim X \equiv 4 \pmod{8}$. This last fact was unknown. Instead of the exterior product representations one has to use the spinor representations.

Let ξ be an $SO(2r)$-bundle with $w_2(\xi) = 0$. Then ξ comes from a $Spin(2r)$-bundle ξ' which we can extend by the right and left spinor representations Δ^+, Δ^- to find two unitary bundles $\Delta^+(\xi')$ and $\Delta^-(\xi')$. Introducing I-equivalence classes we can form the difference

$$\eta = \Delta^+(\xi') - \Delta^-(\xi').$$

Write the Pontryagin classes of ξ as elementary symmetric functions in the squares of certain variables a_i such that the product of the a_i is the Euler class of ξ. Then it follows from the character formula of the spinor representations that

$$\text{(1)} \qquad \mathrm{ch}(\eta) = \prod(\exp(a_i/2) - \exp(-a_i)/2).$$

(The 0-dimensional term in $\mathrm{ch}(\eta)$ is irrelevant for the following: thus I do not bother).

The sphere bundle $\xi^{\hat{}}$ associated to ξ is the same as the sphere bundle associated to ξ'. If one lifts η to the total space $E_{\xi^\wedge}$ one gets the trivial I-equivalence class since the representations Δ^+ and Δ^- become equivalent when restricted to $Spin(2r-1)$. Therefore η gives rise to an I-equivalence class η of unitary bundles (not necessarily uniquely determined) over the Thom space $M(\xi)$ = (mapping cylinder of $E_{\xi^\wedge} \to B$ with E shrunk to a point).

Now let X be a compact oriented differentiable manifold, with $\dim X = 4k$ and $w_2(X) = 0$. Imbed X in a sphere of dim $4k+8n$. Let ξ be the normal bundle ($B_\xi = X$). Then $w_2(\xi) = 0$. The bundle η mentioned before fives rise to a unitary bundle η over S^{4k+8n}. Let φ_* be the Gysin homomorphism $H^*(X) \to H^*(S^{4k+8n})$. Then formula (1) *which is valid in the universal case* yields

$$(*) \qquad \varphi_* \hat{A}(x) = \mathrm{ch}(\eta).$$

Here $\hat{A}(x)$ is of course the "total class" belonging to the power series

$$\tfrac{1}{2}\sqrt{z}/\sinh\tfrac{1}{2}\sqrt{z}$$

Formula (*) corresponds in your proof of the integrality of Todd to the "Grothendieck formula"

4. Letter Hirzebruch → Atiyah dated October 7 1958 (continued).

$\varphi_* T(X) = \mathrm{ch}(\eta)$ where the η comes in this case from the alternating sum of the exterior representations and ξ is a unitary bundle as in Milnor's paper on the complex analogue of cobordism.

The $U(2^{4n-1})$-bundles $\Delta^+(\xi')$ and $\Delta^-(\xi')$ can be reduced to $SO(2^{4n-1})$ by a theorem of E. Cartan and Malcev (see the paper with Borel at the end of 26.5; this was the reason why I chose the codimension of X in sphere to be divisible by 8). It follows easily that η is also the complexification of an orthogonal bundle. Thus according to Bott-Kervaire

(**) $\mathrm{ch}(\eta)[S^{4k+8n}]$ is an integer and an even integer if k is odd.

The formula (*) implies : If $w_2 = 0$, then $\hat{A}(X)$ is an integer ; if moreover $\dim X \equiv 4 \bmod 8$, then $\hat{A}(X)$ is an even integer. For $\dim X = 4$, this is just Rohlin's theorem.

Do you have a mimeographed copy of Milnor's talk in Edinburgh? If not, let me know, and I will send you one. This talk has become a joint paper of Milnor-Kervaire. As you know they prove in there that the image of the stable group $\pi_{4n-1}(\mathrm{SO}(m))$ in the stable group $\pi_{m+4n-1}(S^m)$ under the Whitehead-homomorphisms J is cyclic of a finite order divisible by the denominator of the rational number $B_n a_n/4n$, where B_n is the Bernoulli number and a_n equal 2 for n odd and 1 for n even. This can now be improved by means of (**) to the effect that this order is divisible by the denominator of $B_n/4n$.

P.S. It seems possible that the formula (*) can be used to develop a “Grothendieck theory” for differentiable manifolds.

5. Letter Atiyah → Hirzebruch dated March 13 1959.

19 Beaumont Road,
Cambridge.

13th March 1959.

Dear Fritz,

Many thanks for your various communications, written, typed, mimeographed and oral! I feel it is now my turn to reply. First let me give my official acceptance to your invitation to the "Arbeitstagung". The dates you suggested are quite suitable for me, but I would be prepared to come at other times if this was more convenient for others. However I cannot get away form Cambridge before about June 20 because of examinations. You have a fine invitation list and I hope all will be able to come.

I received your Bourbaki talk all right – in fact I received two copies. You made a nice job of it, and it will make a very useful draft for further versions. One small point about references – you refer at one stage to Puppe for the proof of the appropriate exact sequence for maps into an H-space. I myself have been using the terminology of Hilton and Eckmann (Comptes Rendus notes 1958 P. 2444-2555), which is very elegant and suitable for this purpose.

About the spectral sequence relating $K(X)$ and $H(X)$, I had myself made the same discovery but not using the axioms of cohomology. Of course it amounts to the same thing, but I got directly to the spectral sequence by taking a cell-complex X, defining $H(p, q) = K(X^p, X^q)$ where X^p is the p-skeleton of X, and using the approach to spectral sequences given in Eilenberg and Cartan Chap XV §7. As you say the spectral sequence becomes trivial after tensoring with Ð, and is actually trivial over Ù if there is no torsion or if there is only even-dimensional cohomology. This makes the computation of $K(X)$ for many homogeneous spaces X very simple. Thus if G is without torsion and U is of maximal rank, and if $H(G/U$, Ù$)$ is given by Borel in the usual form in terms of $x_1, \ldots, x_n$, then $\mathrm{ch}(G/U)$ is simply obtained by replacing x_i by e^{xi}. For example for $X = P_n($Æ$)$ $\mathrm{ch}(X) =$ Ù$[e^x]$, where x is the generator of $H^2(X$, Ù$)$. I had remarked this some time ago, and made use of it in connection with the problem of James that I wrote to you about letter of 7 October, but the use of the spectral sequence gives a much more elegant proof. I also agree about writing K^{-q} instead of K^q.

I have now had a provisional reply from Bott to my various questions. So far he has been able to show directly from his maps that his map $S^2\times$Ù$\times B_U \longrightarrow$ Ù$\times B_U$ is given by tensor product with the appropriate element of $K(S^2)$. Actually he had a little difficulty but this was because he had not properly taken into account the augmentation. He has also definite ideas on how to extend this to the real case, and this will then answer all our questions very nicely. This will obviate the indirect proof I had to give to get round the problem of commutativity between the real and complex cases.

I have just found a nice application of the "unstable" Riemann-Roch which I mentioned in my letter of 28 December. The argument on page 3 of that letter which I used to prove the integrality of the A-genus can just as well be used to deduce:

5. Letter Atiyah → Hirzebruch dated March 13 1959 (continued).

Theorem *Let X be a compact oriented differentiable manifold of dimension* $m \equiv 0$ *mod* 4, a*nd suppose that X can be differentiably embedded in Euclidean space of dimension* $2m–2q$. *Then the A–genus of X is divisible by* 2^q.

As usual one can improve this slightly; *if* $q \equiv 2$ *mod* 4 *then* $A(X)$ *is divisible by* $2q+1$.

Of course for Spin manifolds this Theorem tell us nothing. In general however it is quite a good result for problems of embeddability. For example consider the case of $X = P_n$(Æ) where $n=2k$. One deduces P_n(Æ) cannot be embedded in space of dimension $4n–2\alpha(n)–2$, where $\alpha(n)$ is defined as follows: write $n = \sum a_r 2^r$ in binary form, (each $a_r = 0$ or 1), then $\alpha(n) = \sum a_r$. If moreover $\alpha(n) \equiv 2$ mod 4 then P_n Ë Ñ$^{4n-2\alpha(n)}$. Although this result is not always bets possible it is much superior to other known results for *almost all n*. I have tried hard to improve these results by a further factor of 2, bur so far without success. Is it incidentally true that the *A*-genus is always divisible by 2? It certainly is in the first few cases.

Your argument for proving Milnor's result on the divisibility of $s(M)$ directly is very nice. Frank Adams gave me the outline of the proof and I shall now study your manuscript for the details.

I have been trying hard recently to see if I could understand the real reason why the Todd polynomials came into your formula for the Steenrod powers on almost complex manifold. I think I see my way, and if I can get anything interesting I will let you know. It has always seemed to me a great mystery that the Todd polynomials should appear in this context – but perhaps you have some insight into this?

What do you think we should do about writing up and publishing Riemann-Roch and its applications? Do you think it would be appropriate if we published jointly a note in the Bulletin of the American Mathematical Society? This would have to be a brief statement of results with the barest outline of method of proof. We could then take our time thinking about the form of proper publication.

Looking forward to our meeting in Bonn.
Regards to the family,

Your friend,
Michael

P.S. Perhaps this year I may be excused from giving a general colloquium talk.

6. Letter Hirzebruch → Atiyah dated March 22 1959.

22, März1959

Dear Michael:

Thank you very much for your letter of March 13. Here a few mathematical remarks.

1) *The A-genus is always even*. Proof: The characteristic power series is $2\sqrt{z}/\sinh 2\sqrt{z}$ and the coefficient of z^k ,$k \geq 1$ in this power series is $(-1)^k 2^{2k+1}(2^{2k-1}-1)B_k/(2k)!$, as mentioned in Borel-Hirzebruch 25.1. This coefficient does not contain 2 in its denominator, since $(2k)!$ contains at most the power 2^{2k-1} and B_k has 2 exactly to the first power in its denominator. Because of the factor 2^{2k+1} we see that the coefficient is 0 mod 2. Thus the whole A-sequence is trivial mod 2 since its characteristic series mod 2 equals 1. ---- I do not know in general by which power $2^{\mu(k)}$ of 2 the polynomial A_k is divisible ($\mu(k) \leq \alpha(k)$ – See your letter. - See page 6 [= page 13] of this resent letter.) If k is a power of 2, then A_k is exactly divisible by 2. Of course, the fact that the A-genus is an even integer does not contain more information about Pontryagin numbers than the statement that the A-genus is an integer. (There are no relations between Pontryagin numbers mod 2.)

2) I like your application of the A-genus to imbedding problems very much. Perhaps you have already realized that your method gives a more general theorem. For an almost complex manifold X and an element ξ Î $K(X)$ the number $T(X, \xi)$ is defined. For the definition of this number one uses only the first Chern class, the Pontryagin classes of X, and the Chern character of ξ. Replace in this definition c_1 by an arbitrary 2-dimensional integral cohomology class d of X and $\mathrm{ch}(\xi) = e^{x1} + e^{x2} + \cdots + e^{xq}$ by $e^{x1/2} + e^{x2/2} + \cdots + e^{xq/2}$. Having no suitable terminology in the moment, let me denote this multiplicative Chern character by $\mathrm{ch}(\xi^{1/2})$. Then the number obtained from $T(X, \xi)$ by the manipulations just described shall be denoted by $\hat{A}(X, d/2, \xi^{1/2})$. This notation is in accordance with Borel-Hirzebruch 25.5 Of course, if α is the line bundle with characteristic class d then

(1) $$\hat{A}(X, d/2, \xi^{1/2}) = \hat{A}(X, 0, (\xi Ä \alpha)^{1/2}).$$

The number $\hat{A}(X, d/2, \xi^{1/2})$ is defined for any compact oriented differentiable manifold X and any ξ Î $K(X)$.

Theorem: *If dim X= $2n$ and if X is imbeddable in Ñ$^{2n+2k}$, then*

$$2^{n+k}\hat{A}(X, d/2, \xi^{1/2}) \textit{ is an integer.}$$

This theorem tells us nothing if $d \equiv w_2$ mod 2 and $\mathrm{ch}(\xi^{1/2})$ Î $\mathrm{ch}(K(X))$, because then $\hat{A}(X, d/2, \xi^{1/2})$ is the value of an element of the Riemann-Roch group on the fundamental cycle of X.

The proof of the above theorem is by your standard method.

Suppose X Î Ñ$^{2n+2k}$. Let $y_1, \ldots, y_k$ be the formal roots of the normal bundle and $x_1, \ldots, x_q$ the formal roots of ξ. Then

$$æ((e^{x1} + \cdots + e^{xq})(\textstyle\prod_{j=1}^{k}(e^{yi} - e^{-yj}))/y_1 y_2 \cdots y_k)$$

is an integer. ($æ$: taking the value on X). For the moment, we denote this integer by a. We have

6. Letter Hirzebruch → Atiyah dated March 22 1959 (continued).

$$2^{-k}\cdot a = æ((e^{x1} + \cdots + e^{xq})\textstyle\prod_{j=1}^{k}(\sinh y_j/y_j))$$

and

$$2^{-n-k}\cdot a = æ((e^{x1/2} + e^{x2/2} + \cdots + e^{xq/2})\textstyle\prod_{j=1}^{k}(\sinh y_j/2)/(y_j/2))\hat{A}(X, 0, \xi^{1/2}).$$

Because of (1) the theorem is proved.

The theorem admits many applications.

If X is an algebraic variety of complex dimension n and H a divisor on X, then we can consider Hilbert's polynomial

$$P(x) = \chi(X,xH).$$

If we take in the theorem for d the first Chern class of X and for ξ the line bundle belonging to the divisor H, we get the following result:

If X is imbeddable in Ñ$^{2n+2k}$, then $2^{n+k}P(x)$ is an integer for every half-integer x.

If for example $P(x)$ has the form

$$P(x) = (a/n!)(x+c_1)(x+c_2)\cdots(x+c_n) \tag{2}$$

with integers c_i, a being the intersection number $H\circ H\circ\cdots\circ H$ (n times), then $2^{n+k}P(1/2)$ has to be an integer. This means that $2^k(a/n!)$ does not contain 2 in its denominator.

(2) is correct if X is one of the irreducible hermitian symmetric spaces $U(p+q)/U(p)\times U(q)$, $SO(2p)/U(p)$, $SO(2p+2)/SO(2p)\times SO(2)$, $E6/Spin(10)\times T^1$, $E7/E6\times T^1$ and if H corresponds to the generator of $H^2(X,$ Ù$)$. This generalizes your result on the complex projective spaces. For these hermitian symmetric spaces the result gets worse corresponding to the power of 2 contained in the degree of the projective imbedding corresponding to H. For example, the quadric of complex dimension $n = 2p$ cannot be imbedded in the space of dimension 4n – $2\alpha(n) - 4$. The degree of the imbedding of $E6/Spin(10)\times T^1$ is 78. This space has complex dimension 16. Therefore it cannot be imbedded in Ñ^{58}. The space $E7/E6\times T^1$ has dimension 27. The degree of its projective imbedding is 13110. Therefore it cannot be imbedded in Ñ^{96}. (Compare my Princeton talk, Characteristic numbers of homogeneous domains). I did not try yet to study the possible applications more systematically, but at least I applied the theorem of page 2 [= page 11] to the quaternionic projective spaces $P_q(K) = Sp(q+1)/Sp(1)\times Sp(q)$. As usual write its cohomology ring in the form Ù$[x_1^2]$. Then $e^{x1} + e^{-x1}$ is the Chern character of an unitary bundle ξ, the unitary extension of the canonical $Sp(1)$-bundle over $P_q(K) = X$. We have to calculate the number $\hat{A}(X, 0, \xi^{1/2})$. Because of the known relationship between the Pontryagin classes of X and those of the quadric of complex dimension $2q$ which we denote here by Y we get

(3) $\quad 2\cdot\hat{A}(X, 0, \xi^{1/2}) = 2\cdot\chi(Y,(-q+1/2)H)$ $\quad$ (= $2T(Y,(-q+1/2)g)$, g point, generator of $H^2(X,$ Ù$)$.)

where H is the hyperplane section of the quadric. The 2 on the left side of this equation comes form the fact that $H\circ H\circ\cdots\circ H$ ($2q$ times) whereas $x_1^{2q} = 1\cdot$generator. The 2 on the right side of the equation comes, if one likes, form Serre duality. Since $\chi(Y, xH)$ is a polynomial P in x of the form (2) with

6. Letter Hirzebruch → Atiyah dated March 22 1959 (continued).

$a = 2$ and since by (3) $\hat{A}(X, 0, \xi^{1/2}) = P(-q+1/2)$ we conclude from $P_q(K) \in \mathbb{R}^{4q+2k}$ that $2^{k+1}/(2q)!$ does not contain 2 in its denominator. In other words, *$P_q(K)$ cannot be imbedded in $\mathbb{R}^{8q-2\alpha(q)-4}$*.

Perhaps one should be able to prove that $P_q(K)$ cannot be imbedded in $\mathbb{R}^{8q-2\alpha(q)-2}$, but I do not see right now how this could be obtained. If q is a power of 2, then it is known (elementary argument using Stiefel-Whitney classes) that $P_q(K)$ cannot be imbedded in $\mathbb{R}^{8q-2}$.

3) I have trouble with your remark that your imbedding theorem can be improved by a factor 2 in certain cases. To get this, it seems necessary that your element η of the representation ring of $SO(2m)$ (letter of December 28, page 3) comes for m even from a "virtual" orthogonal representation. Is this clear? If yes, also the theorem on page 2 [= page 11] of my present letter could be improved in certain cases.

4) I will think about your question concerning the Todd polynomials in connection with cohomology operations. Here only two preliminary remarks.

a) The occurrence of Todd polynomials in this context can be motivated by the theorem that $T(X, \xi)$ is integral. If X has complex dimension p and if ξ is a line bundle with first Chern class d (p prime, $d \in H^2(X, \mathbb{Z})$), then

(4) $$d^p/p! + (d^{p-1}/(p-1)!)T_1 + \cdots + dT_{p-1} + T_p \qquad \text{integral.}$$

Since T_p is integral too, we get from (4) by multiplication with p that

(5) $$d^p/(p-1)! + d(pT_{p-1}) \equiv 0 \mod p,$$

here one uses that T_{p-1} is the first Todd polynomial containing p (namely exactly to the first power) in its denominator. Since $(p-1)! \equiv -1 \mod p$, we get from (5) $d^p \equiv d(pT_{p-1}) \mod p$.

A similar calculation is probably possible if one replaces d by the class dual to a subvariety Y and ξ by $i_!(1)$. (i: $Y \to X$ injection). Since in this case d is a Chern class, the effect of the Steenrod powers on d is probably hidden in the Chern character of ξ. I shall try the calculation, but perhaps you did just the same.

b) The relations à la Wu between Pontryagin numbers can also be obtained from an imbedding of the manifold in a sphere. This is formally similar to *RR*. It yields the proof that à la Wu one gets all relations between Pontryagin numbers (compare Dold, Math, Zeitschrift 65 (1956), who did the same for Stiefel-Whitney numbers). I will write the details at some other occasion (continuation of my exposé: Cohomologie-Operatonen in Mannigfaltigkeiten).

5) This is an appendix to 1) on page 1 [= page 11]. I have just found a proof that $\mu(k) = \alpha(k)$. Recall that $2^{\mu(k)}$ is defined as the largest power of 2 such that $A_k 2^{-\mu(k)}$ is a polynomial whose coefficients do not contain 2 in their denominators. $\alpha(k)$ is defined as in your letter.

First one calculates the A-genus of $P_{2k}(\mathbb{C})$. We get

$$A(P_{2k}(\mathbb{C})) = (2k)!/k!k! \;\; \text{which is precisely divisible by } 2^{\alpha(k)}.$$

6. Letter Hirzebruch → Atiyah dated March 22 1959 (continued).

Every oriented compact differentiable manifold of dimension divisible by 4 can be written as polynomial in the $P_{2k}(\hat{A})$. The coefficients of this polynomial have no 2 in their denominators. (The determinant of Pontryagin numbers of the products $P_{2j1}\times \cdots \times P_{2jr}$, $j_1 + \cdots + j_r = k$, is odd. This shows also that there are no relations mod 2 between Pontraygin numbers.) The A-genus of $P_{2j1}\times\cdots\times P_{2jr}$ is divisible by 2 to the power $\alpha(j_1) + \alpha(j_2) + \cdots + \alpha(j_r)$ which exponent is not less than $\alpha(k)$. Thus the A-genus of every M^{4k} is divisible by $2^{\alpha(k)}$ and this is also the best possible result.

If $\mu(k)$ would be less than $\alpha(k)$, the just underlined result would give a relation mod 2 between Pontryagin numbers which is impossible. *Thus* $\mu(k) = \alpha(k)$.

This is a purely "number theoretic" result. The above proof is also purely algebraic if one replaces the complex projective spaces by the systems of their Pontryagin numbers.

The result that the $\hat{A}$-genus of a Spin-manifold M^{4k} is an integer (k even) respectively an even integer (k odd) is thus a relation between Pontryagin number modulo $2^{4k-\alpha(k)}$ respectively $2^{4k-\alpha(k)+1}$.

A daring conjecture would be that $2^{4k-\alpha(k)}$ – resp. $2^{4k-\alpha(k)+1}$ – multiple of any manifold is cobounding to a Spin-manifold.

I think I should come to an end. Many thanks for your generous offer of publishing jointly about *RR*. This would be very nice, though I have a little bit a bad conscience since you had the original ideas. But perhaps I can continue to contribute and we have time to work together in Princeton. So if you like, we can start immediately to write the short note for the Bulletin. It should also contain applications to make it more interesting for some more people. How should we do the writing job? Who shall start to write?

I am supposed to give a talk at Lille (Colloque C.R.N.S.) in the beginning of June. I must give them a manuscript. Perhaps I will take the cohomology operations if this is not too trivial. In the moment I am very interested in these imbedding problems. Perhaps one gets nice results for a larger class of algebraic varieties. In the paper, which should only be a few pages, I could report about your theorem on the A-genus, the generalisation on page 2 [= page 11] of this letter and applications to projective spaces and some algebraic manifolds. Perhaps it would be appropriate to make this Lille paper a joint paper, or I could say that I am reporting on your methods like in the case if Bourbaki. We could then omit the imbedding applications from the Bulletin referring to Lille. What do you think?

I am very glad that you will come to the "Arbeitstagung". Adams, yourself and Serre have accepted agreeing to the proposed date. Borel, Grothendieck have accepted, but they do not know for sure yet whether the proposed date will be convenient. Milnor will be probably attending since he is here as a visitor for one month. Thom and James cannot come since they are in Mexico or Chicago respectively.

With my best regards to all the family and a happy Easter
Yours,

Fritz

7. Letter Hirzebruch → Atiyah dated march 28 1959.

March 28, 1959

Dear Michael:

I am trying to get a more general formulation for the non-imbeddability theorems. It is easy to prove

I) *Let X be a compact oriented differentiable manifold of dimension* $2n$ *with Stiefel-Whitney class* $w_3 = 0$. *If there exists an element* d Î $H^2(X,$ Û$)$ *such that* $<d^n, X>$ *is odd, then X is not imbeddable in the space of dimension* $4n-2\alpha(n)-2$.

Proof: Let c_1 be an element of $H^2(X,$ Û$)$ whose restriction mod 2 is w_2. Then $\hat{A}(X, c_1/2, \eta^t)$, where η is the line bundle with cohomology class d, is a polynomial in t of degree n which takes for integral t integral values. It can therefore be written in the form

$$(1) \qquad \hat{A}(X, c_1/2, \eta^t) = a_n\binom{t}{n} + a_{n-1}\binom{t}{n-1} + \cdots + a_1\binom{t}{1} + a_0 = P(t),$$

where the a_j are integers. ($a_n = <d^n, X>$). Assume that X can be imbedded in Ñ$^{2n+2k}$. Then $2^{n+k}P(1/2)$ is an integer. This gives

(2) $2^k(a_n \cdot \text{odd}/n! + 2na_{n-1} \cdot \text{odd}/n! + \cdots + 2^{n-1}n! \cdot \text{odd}/n! + 2^n a_0 n!/n!)$ is an integer. It follows because a_n is odd that

(3) $2^k \cdot \text{odd}/n!$ is an integer which proves the desired result.

"odd" stands always for some odd integer.

II) *Let X be a compact oriented differentiable manifold of dimension* $4m$ *and with* $w_3 = 0$. *If there exists a d* Î $H^2(X,$ Û$)$ *such that* $<d^{2m}, X>/2$ *is an odd integer, then X cannot be imbedded in* Ñ$^{8m-2\alpha(m)-4}$.

For the proof one uses (2) with $n = 2m$. Since $a_n = 2 \cdot \text{odd}$, one deduces that $2^{k+1}\text{odd}/n!$ is an integer if X Î Ñ$^{4m+2k}$, which proves (II).

It is clear that one can find more theorems of a similar nature.

Do you know a compact oriented M^{2n} *which cannot be imbedded in* Ñ$^{4n-2\alpha(n)+2}$?

Can one find a concrete imbedding of $P_n($Â$)$ in Ñ$^{4n-2\alpha(n)+2}$?

James told me something about concrete imbeddings of projective spaces but I forgot the details. Do you know them?

I am very surprised by these strong results concerning non-imbeddability obtained by your method.

Cordial greeting to all of you

7. Letter Hirzebruch → Atiyah dated march 28 1959 (continued).

March 28, 1959

P.S.

I) and II) can be generalized. Let ξ Î $K(X)$ where X is compact oriented differentiable of dimension $2n$. Then we define $s(\xi)$ = æ$(n!\mathrm{ch}(\xi))$. If X is almost-complex and ξ its complex tangent bundle, then $s(\xi)$ is the usual $s(X)$. If η Î $K_0(X)$ and if η_0 is its complex extension, then define $s(\eta) = 1/2s(\eta_0)$; this is 0 if dim $X \equiv 0$ (4) and if η is the real tangent bundle then $s(\eta)$ equals the usual $s(X)$.

III) *Let X be a compact oriented differentiable manifold of dimension $2n$ with Stiefel-Whitney class $w_3 = 0$. If there exists an element ξ Î $K(X)$ such that $s(\xi)$ is odd, then X is not imbeddable in the space of dimension $4n-2\alpha(n)-2$.*

Proof. Write $\mathrm{ch}(\xi) = \sum_{j=0}^{\infty}\mathrm{ch}_j(\xi)$ with $\mathrm{ch}_j(\xi)$ Î $H^{2j}(X,$ Ð$)$. Then for every integer t also $\sum_{j=0}^{\infty}t^j\mathrm{ch}_j(\xi)$ belongs to $\mathrm{ch}(K(X))$ and is the Chern character of a canonical $\xi^{(t)}$ Î $K(X)$. This is a consequenceof the fact that $e^{tx1} + e^{tx2} + \cdots + e^{txq}$ belongs to the representation ring of $U(n)$. Like in I) we have a polynomial $P(t) = \hat{A}(X, c_1/2, \xi^{(t)})$ with $a_n = s(\xi)$. This proves III). In the same way we get

IV) *Let X be as in* III) *but now of dimension $4m$. If there exists ξ Î $K(X)$ such that $s(\xi)/2$ is an odd integer. Then X cannot be imbedded in* Ñ$^{8m-2\alpha(m)-4}$.

This has the result on the quaternionic projective spaces as corollary.

In particular we have:
(*) *Let X be compact oriented differentiable of dimension $4m$. If $s(X)$ is odd, then X cannot be imbedded in* Ñ$^{8m-2\alpha(m)-4}$.

These results are subject to certain improvement (compare my question 3) on page 4 [= page 13] of my preceding letter.

Perhaps it would be worthwhile motivated by (*) to look at the whole business from the cobordism-view-point.

Which X^{4m} are cobounding to a Y^{4m} imbeddable in Ñ$^{8m-d}$ (d given)?

Perhaps the condition $w_3 = 0$ in I – IV can be avoidable.

Appendix: Transcription of the comment page 2 of the letter dated March 22 1959.

Following is a transcription of the comment in German found around "Theorem" on page 2 [= page 11].

ξ Î $K_0(X)$ orthogonal, k even

$2n+2k \equiv 0\ (4) \bmod 8$

$\boxed{n+k \equiv (2) \bmod 4}$

ß

a even

ß

$2^{n+k-1}\hat{A}(\cdots)$ integer.