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HARVARD UNIVERSITY
CAMBRIDGE, MASSACHUSETTS 02138

DEPARTMENT OF MATHEMATICS
2 DIVINITY AVENUE

2 Av. de Verrière !

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Professor Alexander Grothendieck
59 rue du Moulon
Bures-sur-Yvette
Seine-et-Oise, France

Dear Shurik:

Greetings! I hope you and yours are all fine in your Paris castle. Sorry you didn't come to the Arbietstagung.

For some time now I have been puzzled about the description of the Chern-classes in hypercohomology. For characteristic zero I have a nice formula involving curvature, but it is hard to make sense of it in characteristic p. The difficulty I run into is already apparent in the construction of the Chern classes a'la Atiyah in $H^r(X; \Omega^r)$, and it seems to me that your account in one of your exposes - (I am sorry but I only saw it in Oxford in a bound book containing various papers and I don't know the precise reference)- doesn't come to grips with this difficulty.

= De Rham cohomology

i.e. in Hodge cohomology

The point is simply that if $k(E) \in H^1(X, \Omega^1 \otimes \text{End } E)$ denotes the Atiyah curvature of E, and $k^2(E)$ denotes its rth power in $H^r(X, \Omega^r \otimes \text{Sym}^r(\text{End } E))$ then a homogeneous invariant polynomial φ does not induce a map from $\text{Sym}^r(\text{End } E)$ to 0. As I see it such a φ only defines a map from $\Gamma_r(\text{End } E)$ to 0 where $\Gamma_r(\text{End } E) \subset E^{(r)}$ is the subspace of invariant elements under the permutation group. The problem therefore seems to be to define an operation $S^r : H^1(X, \Omega^1 \otimes \text{End } E)$ to $H^r(X, \Omega^r \otimes \Gamma_r(\text{End } E))$ and I don't see how this is to be done.

Your comments on this would be very welcome.

Let me just briefly indicate how the complete Chern-class can be defined in hypercohomology, in the characteristic zero case, say for the base field $\mathbb{C}$.

We start with the bundle E over X, and an invariant polynomial φ of degree r which therefore induces a degree r-map

$$\varphi : \text{End } E \to 0 ,$$

and also denote its obvious extension from

$$\Omega^2 \otimes \mathrm{End}\, E \to \Omega^{2r}$$

by φ. The problem is to define $\varphi(E) \in H^{2r}(X; \Omega)$.

For this purpose let $U = \{U_\alpha\}$ be a cover of X such that E/U_α admits a connection, say $D = \{D_\alpha\}$. Relative to this covering $\varphi(E)$ is then represented by a Cech cochain

$$\varphi(D) \in \sum_{p+q=2r} C^p\{U; \Omega^q\} \qquad \text{as follows:}$$

For each s-simplex σ of the nerve of U, let

$$U_\sigma = U_{\alpha_0} \cap \cdots \cap U_{\alpha_s}.$$

Also let $\mathbb{C}^{s+1}$ be complex $s+1$ space and let $H^s \subset \mathbb{C}^{s+1}$ be the subset $\Sigma_{i=0}^{s} t_i = 1$. We lift E to $U_\sigma \times H^s$ and there define a connection D_σ for E by simply setting

$$D_\sigma = \sum_{i=0}^{s} t_i D_{\alpha_i}.$$

Next let K_σ be the curvature of D_σ in the usual sense, and let $\varphi(K_\sigma)$ be the resulting form on $U_\sigma \times H^s$. Now if we interpret σ as the set t_i real; $0 \le t_i \le 1$; then

$$U_\sigma \times \sigma \subset U_\sigma \times H^s$$

and we may integrate $\varphi(K_\sigma)$ over the fiber σ, to obtain a form

$$\int_\sigma \varphi(K_\sigma) \subset \Omega^{2r-|\sigma|}(U_\sigma).$$

Modulo signs I now claim that the cochain

$$\sigma \longrightarrow \int_\sigma \varphi(K_\sigma)$$

represents $\varphi(E)$.

Remarks. (1) The integration introduced here is ofcourse a purely formal matter, it does involve rational denominators though and hence only makes sense in characteristic zero.

(2) As Segal pointed out to me in Oxford, this construction involves a method he used before, in his K-theory, namely the consideration of the subset

$$\tilde{X} \subset X \times N(U), \qquad N(U) = \text{Nerve of } U$$

consisting of the pairs (x, σ) with $x \in U_\sigma$.

Well so much for now - any comments on this you might have would also be appreciated. In particular do you expect a naive formula like this to hold in characteristic p also ?

Finally, last spring Hartshorne brought back news that you had a very elegant treatment of some of the formulas of Baum and myself. Unfortunately the trip had washed all the details out of his brain. I meant to write you then about all these matters but various events, e.g., our rebellion intervened, I will send you a copy of the opus with Baum, and would be much interested in your account of the matter.

With best regards to all of you.

Yours,

Raoul

Raoul Bott

RB/mfm

P.S. I will be in LaJolla (Department of Math, University of California, San Diego; LaJolla, California)'till August 18th. Thereafter mail would be sent to Harvard.