FG-ILZMIYANISHI67
Massy 9.12.1967

Dear Miyanishi,

I found your letter of Oct.27 a few days ago, after a one month's stay in North Vietnam where I did not get any of my mail; so please excuse if I did not answer before. I did not read more than the statement of results of your paper "On the cohomologies of commutative affine group schemes", without diving into the technicalities of your proofs. The reason for this is that I think a rather simple method should yield more general results. The method is the one suggested on the last two pages of SGA 2 (Sém. Géom. Alg. 1960-61). First a "general non-sense" statement, to put the remarks of SGA 2 alluded to into the right context:

<u>Proposition</u> 1 Let <u>S</u> be a topos (Cf SGA 4 (1963/64) for this notion) $f: X \to S$ a morphism in S, ~~having a section g~~ F an abelian sheaf on S, G an abelian sheaf on X. We then get a canonical morphism

$$R^1 f_*(\underline{Hom}(f^*(F), G)) \longrightarrow \underline{Hom}(H, R^1 f_*(G)) .$$

The latter is an isomorphism in each of the following two cases; if moreover $\underline{Ext}^1(f^*F, G) = 0$:

(i) X/S has a section g, and the canonical morphism

$$f_*(G) \longrightarrow g^*(G)$$

is an isomorphism.

(ii) G is of the form $f^*(G_0)$, with G_0 an abelian sheaf on S such that the canonical morphism

$$G_0 \to f_*(f^*(G_0)))$$

is an isomorphism, and f is an epimorphism.

For the proof, one can first deal with case (i) by a rigidification technique, interpreting (using $\underline{Ext}^1(H, \mathfrak{G}) = 0$) a torsor (=principal hom. sheaf) under a sheaf $\underline{Hom}(H,G)$ as a torsor on H, with group $\mathfrak{G}_H$, endowed with some extra structure which intuitively can be expressed by saying that we have a group homomorphism of H with values in the ~~category~~ group-like category of G-torsors (with the usual "product" of two torsors under the abelian structure group G). On the other hand, (ii) is a corollary of (i), because we reduce to case (i) using the base change $S'=X \to S$. – I shou[ld] confess I did not really write out the proof of the proposition, so mayb[e] I forgot some mild extra assumption. The point is that if we now take a proper and flat morphism of finite presentation of schemes

$$f:X \longrightarrow S,$$

such that $f_*(\underline{O}_X) \xleftarrow{\simeq} \underline{O}_S$ universally, one should be able to apply the general non-sense proposition to the topi of fppf sheaves on X,S (fppf = fidèlement plat de présentation finie), and the sheaf $G_o = \underline{G}_{m\ S}$ on S; note that the assumption on f implies the assumption in (ii), on the other hand $R^1f_*(G) = R^1f_*(\underline{G}_{m\ X}) = \underline{\mathrm{Pic}}_{X/S}$, the relative Picard shea[f] of X over S. The formula becomes (assuming $\underline{\mathrm{Ext}}^1(f^*(F), \underline{G}_{m\ X}) = 0$) :

(1) $$\cancel{R^1f_*(\underline{Hom}(f_*(F),G}\quad R^1f_*(f^*(D(F))) \simeq \underline{\mathrm{Hom}}(F,\underline{\mathrm{Pic}}_{X/S}) \quad ,$$

where for any abelian sheaf E on X, $D_X(E)$ is its Cartier dual

$$D_X(E) = \underline{\mathrm{Hom}}(E,\underline{G}_{m\ X}) \quad .$$

Therefore, for a sheaf H on S isomorphic to a sheaf D(F), the representability of $R^1f_*(f^*(H))$ is equivalent to the representability of $\underline{\mathrm{Hom}}(F,\underline{\mathrm{Pic}}_{X/S})$, a rather standard kind of problem if we assume already $\underline{\mathrm{Pic}}_{X/S}$ itself representable. Let us consider two particular cases:

a) H is finite, locally free over S. By Cartier duality, we have

$$H = D(F) \ , \text{ where } \ F = D(H) \quad ,$$

F being itself finite and locally free over S. We now want to represent $\underline{\mathrm{Hom}}(F,\underline{\mathrm{Pic}}_{X/S})$; this is clearly possible if we assume that $\underline{\mathrm{Pic}}_{X/S}$ is representable, and moreover, either that every finite subset of a fibre is contained in an affine open subset of $\underline{\mathrm{Pic}}_{X/S}$ (this is OK if we assume f projective with integral fibers, as is well known), or if F is radicial over S.

b) $S = \mathrm{Spec}(k)$, k a field. Then any commutative affine algebraic group scheme H over k is of the type $D(F)$, where F is a sheaf which in general does not correspond to an algebraic group, but is a "formal group" defined by the affine algebra of H as its hyperalgebra. For instance the dual of $\underline{G}_m$ is the constant group $\underline{Z}$, the dual of $\underline{G}_a$ corresponds to the enveloping algebra of the abelian Lie algebra generated by one element T. It is certainly standard to check that for any commutative group scheme P locally of finite type over k, $\underline{\mathrm{Hom}}(D(F),P)$ is representable. In the two typical cases above, it is representable ~~either~~ by P itself in the case $H=\underline{G}_m$, by the Lie algebra of P (or rather the linear variety defined by it) in the case $H=\underline{G}_a$. To check representability in general, we may by radicial flat descent reduce to the case when k is perfect, and hence H decomposes into $H_u \times H_m$, with H_u unipotent and H_m of multiplicative type, and then deal separately with these two cases. The case of multiplicative type is immediate (reduce to H diagonalisable by finite descent). The case H unipotent will demand more care, but should not offer any difficulty. One should also be able to express explicitly the scheme representing $\underline{\mathrm{Hom}}(D(H),P)$ in terms of the Dieudonné modules associated to H and to the formal group defined by P .... (NB D(H) is purely infinitesimal); there are results along these lines in Oda's thesis, I believe.

In case a), assuming moreover that X is an abelian scheme over S, and denoting by X' the dual abelian scheme (NB it is proved that the Picard functor of any abelian scheme is representable ...), we get that $\underline{\mathrm{Hom}}(F,\underline{\mathrm{Pic}}_{X/S}) \simeq \underline{\mathrm{Hom}}(F,X')$. Interpreting à la Serre X' as being the sheaf $\underline{\mathrm{Ext}}^1(X, \underline{G}_{mS})$, we get the result that for a commutative finite locally free group scheme H over S, the natural morphism

$$(2) \qquad \underline{\mathrm{Ext}}^1(X,H) \longrightarrow R^1 f_*(f^*(H))$$

is an isomorphism, or what amounts to the same, the map

$$\mathrm{Ext}^1(X,H) \longrightarrow H^0(S,\ R^1 f_*(f^*(H))$$

is an isomorphism. This is one of the results you state in your paper on coverings of abelian varieties (when $S = \mathrm{Spec}\,k$). The same will hold true if S=Spec(k), and if H is an (affine commutative) algebraic group scheme such that $(H_{\bar{k}})^0_{red}$ is ~~xxxxxx~~ unipotent.

In the previous statements, the affineness assumption on H seems rather natural, as it allows rigidification techniques, using that $H \simeq f_*(f^*(H))$; however it seems worthwhile to look if one gets also representability theorems without that assumption. Another question, assuming say H to be ~~affine~~ (finite over S), is to get results without commutativity assumption for H. For instance, if H is ~~finite~~ étale, one can give a rather explicit criterion for $R^1 f_*(f^*(H))$ to be representable by a scheme étale and separated over S. It seems to me an interesting question to prove a result in this direction, without assuming H étale (but only locally free say). Also, I certainly expect the isomorphism (2) still to hold. In case S=Spec(k), this would be expressed equivalently by saying that the natural homomorphism

$$\boxed{\varprojlim_n {}_nX \longrightarrow \pi(X,e)},$$

where for every integer $n \geq 1$, ${}_nX = \mathrm{Ker}\ n\ \mathrm{id}_X$, and where $\pi(X,e)$ is the "true fundmental group of X at the point e", classifying torsors over X with finite algebraic group-schemes as structure group and with a rigidification given over e (i.e. a point rational over k chosen over) is an isomorphism.
This latter statement would be the really satisfactory formulation of

various partial results you state concerning coverings of abelian varieties. By standard arguments, the fact that (2) is an isomorphism would follow if we knew that the right hand member is ~~additive~~ "multiplicative with respect to X." This in turn reduces readily to the following

<u>Conjecture</u> Let $f: X \longrightarrow S$ and $g: Y \longrightarrow S$ be proper, flat, of finite presentation, with $\underline{O}_S \xrightarrow{\sim} f_*(\underline{O}_X), g_*(\underline{O}_Y)$ universally, and let f', g' be sections of X/S and Y/S, let H be a ~~finite~~ group scheme over S, finite and of finite presentation and flat, P a torsor over $X \times_S Y$ with group $H_{X \times_S Y}$. Assume given trivialisations of P over $S \times_S Y$ and over $X \times_S S$ which agree on $S \times_S S$; then there exists a trivialisation of P (i.e. a section) inducing the preceding ones. (NB This trivialisation is necessarilyn uniqqe).

Taking S to be the spectrum of a field k, this conjecture amounts essentially to the conjecture that (if a,b are the rational points corresponding to f',g') the canonical homomorphism of profinite group schemes

$$\pi(X \times Y, (a,b)) \longrightarrow \pi(X,a) \times \pi(Y,b)$$

is an isomorphism. THe general conjecture and this special ~~fact~~ consequence, seem to me extremely plausible, and a ppoof of it very desirable. The case H étale is of course known (SGA 1)

Sincerely yours

A Grothendieck

PS In the cases where I state that formula (1) can be applied, one has of course to check $\underline{\text{Ext}}^1(f^*(F), \underline{G}_{m\,X}) = 0$. Here an assumption on F, which affinness ~~of~~ over S (F an abelian variety wouldn't do). certqinly enters in an essential way, Heuristically, when Cartier duality is applicable, this reduces fo $\underline{\text{Ext}}^1(D(\underline{G}_{mX}), D(f^*(F))) = 0$, i.e. to $\underline{\text{Ext}}^1(\underline{Z}_X, f^*(H)) = 0$, which is indeed true. To make this argument applicable in case a) (and not only case b)) one has to write down with some care a statement of Cartier duality, for some kind of ~~affine~~

affine commutative group schemes on the one hand, duals which are kind of formal group schemes, and extensions of such. This should not offer serious difficulty as long as one is content with very stringent assumptions (such as local freeness of affine algebras and hyperalgebras involved ...). Fo the general idea, see Cartier's talk at ~~xxx~~Bruxelles. Of course, formula (1) will be applicable also for groups such as $\underline{G}_m$, hence $H = \underline{Z}_S$, and we then get the amusing result

$$R^1 f_*(\underline{Z}_X) \simeq \underline{\mathrm{Hom}}_S(\underline{G}_{mS}, \underline{\mathrm{Pic}}_{X/S}) ,$$

which for S=Spec(k), k algebraically closed field, yields

$$H^1(X, \underline{Z}_X) = \mathrm{Hom}(\underline{G}_{mk}, \underline{\mathrm{Pic}}_{X/k}) ,$$

which gives an interpretation of the dimension of the maximal torus of $\underline{\mathrm{Pic}}_{X/S}$ in topological terms. The first hand side is a cohomology group for the fppf topology, or what amounts to the same since $\underline{Z}_X$ is smooth over X, for the etale topology. It's rank is bounded by (but rarely equal to) the first ℓ-adic Betti number of X, where ℓ is a prime $\neq$ char k . By the way, these results should hold without the assumption $\underline{O}_S \simeq f_*(\underline{O}_X)$ universally; indeed, for the validity of (1), I guess that this assumption could be relaxed to the assumption of cohomological flatness of f in dimension 0, i.e. $f_*(\underline{O}_X)$ flat over S and commutes with base change. This would imply that if S=Spec(k), k a field, X proper over k and nothing more, then for any affine algebraic commutative group scheme H over k, $R^1 f_*(f^*(H))$ is representable.

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