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Bures May 11, 1970

Dear Barsotti,

I would like to tell you about a result on specialization of Barsotti-Tate groups (the so-called p-divisible groups in Tate's terminology) in char.p , which perhaps you know for a long time, and a corresponding conjecture or rather question, whose answer may equally be known to you.

First some terminology. Let k a perfect field of char $p>0$, W the ring of Witt vectors over k, K its field of fractions. An F-<u>cristal over</u> k will mean here a free module of finite type M over W, together with a σ-linear endomorphism $F_M : M \longrightarrow M$ (where $\sigma : W \longrightarrow W$ is the Frobenius automorphism) such that ~~F(M)xxxxxof~~ F_M is injective i.e. F(M) contains $p^n M$ for some $n \geq 0$. I am rather interested in F-<u>iso</u>-cristals, namely F-cristals up to isogeny, which can be interpreted as finite dimensional vector spaces E over K, together with a σ-linear automorphism $F_E : E \longrightarrow E$, such that there exists a "lattice" $M \subset E$ ~~inner~~ mapped into itself by F_E ; I will rather call such objets <u>effective</u> F-isocristals (and drop the ~~suffix~~ "iso" (and even F) when the context allows it), and consider the larger category of (E, F_E), with no assumption of existence of a stable lattice M made, as the catégory of F-isocristals. It is obtained ~~formally~~ from the category of effective F-isocristals and its natural internal tensor product, by "inverting" formally the "Tate cristal" $K(-1) = (K, F_{K(-1)} = p\sigma)$: the ~~effe~~ isocristals (E, F_E) such that $(E, p^n F_E)$ is effective (i.e. the set of iterates of $(p^n F_E)$ ~~remains~~ is bounded for the natural norm structure) can be viewed as those of the form $E_o(n) = E_o \otimes K(-1)^{\otimes(-n)}$, with E_o an effective F-(iso)cristal.

Assume now k alg. closed. Then by Dieudonné's classification theorem as reported on in Manin's report, the category of F-(iso)cristals over k is semi-simple, and the isomorphism classes of simple elements of this category can be indexed by $\underline{Q}$ (the group of rational numbers), or what amounts to the same, by pairs of relatively prime integers

$$r,s\in\underline{Z}\ ,\ r\geq 1\ ,\ (s,r)=1$$

to such a pair corresponding the simple object

$$E_{s/r} = E_{r,s}$$

whose rank is r , and which for $s\geq 0$ can be described by the cristal over the prime field $\underline{F}_p$ as

$$E_{s/r} = \underline{Q}_p[T]/(T^r-p^s)\quad ,\quad F_{s/r} = \text{multiplication by } T\ .$$

For $s\leq 0$, we get $E_{s/r}$ by the formula

$$E_{-\lambda} = (E_\lambda)^\vee\ ,$$

where $^\vee$ denotes ordinary dual endowed with the contragredient F automorphism. In Manin's report, only effective F-cristals are considered, with the extra restriction that F_E is topologically nilpotent, but by Tate twist this implies the result as I state it now. Indexing by $\underline{Q}$ rather than by pairs (s,r) has the advantage that we have the simple formula

$$E_\lambda\otimes E_{\lambda'}\simeq \text{sum of cristals } E_{\lambda\lambda'}\ .$$

In other words, if we decompose each cristals in its isotypic component corresponding to the various "slopes" $\lambda\in\underline{Q}$, so that we get a natural graduation on it with group $\underline{Q}$, we see that this graduation is compatible with the tensor product structure:

$$\cancel{E\otimes E}\quad E(\lambda)\otimes E'(\lambda')\subset (E\otimes E')(\lambda+\lambda')\ .$$

The terminology of "slope" of an isotypic cristal, and of the sequence of slopes occuring in any cristal (when decomposing it into its isotyp components) is due, I believe, to you, as we discussed on formal groups in Pisa about three years ago; but I did not appreciate then the full appropriateness of the notion and of the terminology. Let's define the <u>sequence of slopes</u> of a cristal (E, F_E) by its isotypic decomposition, repeating each λ a number of times equal to rank $E(\lambda)$ (bearing in mind that if $\lambda = s/r$ with $(s,r)=1$, then the multiplicity of λ in E i.e. rank $E(\lambda)$ is a multiple of r); moreover it is convenient to order this sequence in increasing order. This definition makes still a good sense if k is not algebraically closed, by passing over to the algebraic closure of k; in fact, the isotypic decomposition over $\bar{k}$ <u>descends</u> to k, so we get much better than just a pale sequence of slopes, but even a canonical "iso-slope" ("isopentique" in french) decomposition over k

$$E = \bigoplus_{\lambda \in \mathbb{Q}} E(\lambda) \quad .$$

(NB This is true only because we assumed k perfect; there is a reasonable notion of F-cristal also if k is not perfect, but then we should get only a <u>filtration</u> of a cristal by increasing slopes ...). Now if k is a finite field with q elements, ~~of rank~~ a over the prime field, and if (E, F_E) is a cristal over k, then ${F_E}^a$ is a linear endomorphism of E over K, and it turns out that the slopes of the cristal are just the valuations of the proper values of ${F_E}^a$, for a valuation of $\bar{\mathbb{Q}}_p$ normalised in such a way that

$$v(q) = 1 \text{ , i.e. } v(p) = 1/a \quad .$$

(This is essentially the "technical lemma" in Manin's report, the restrictive conditions in Manin being in fact not necessary.) Thus, the sequence of slopes of the cristal, as defined above, is just the sequence of slopes of the <u>Newton polygon</u> of the characteristic polynomial of the arithmetic Frobenius endomorphism ${F_E}^a$, and their knowledge is equivalent to the knowledge of the p-adic valuations of the proper values of this Frobenius !

Lets come back to a general perfect k. Then the cristals which are ~~Dieudonné modules~~ effective are those whose slopes are > 0; those which are Dieudonné modules, i.e. which correspond to Barsotti-Tate groups over k (not necessarily connected) are those ~~of slope~~ whose slopes are in the closed interval $[0,1]$: slope zero corresponds to ~~(a twist of)~~ ind-étale groups, slope one to toroidal groups. Moreover, an arbitrary cristal decomposes canonically into a direct sum

$$E = \bigoplus_{i \in \underline{Z}} E_i(-i) \quad ,$$

where $(-i)$ are Tate twists (corresponding to multiplying the F endomorphism by p^i), and the E_i have slopes $0 \leq \lambda < 1$ (or, if we prefer, $0 < \lambda \leq 1$), and hence correspond to Barsotti-Tate groups up to isogeny over k, without toroidal component (resp. which are connected). The interest of this remark comes from the fact that if X is a proper and smooth scheme over k, then the cristallin cohomology groups $H^i(X)$ can be viewed as F-cristals, H^i with slopes between 0 and i(*) and define in this way a whole avalanche of Barsotti-Tate groups over ~~k which~~ (up to isogeny), which are quite remarkable invariants whose knowledge should be thought as essentially equivalent with the knowledge of the characteristic polynomials of the "arithmetic" Frobenius acting on (any reasonable) cohomology of X (although the arithmetic Frobenius is not really defined, unless k is finite !).

Now the result about specialization of Barsotti-Tate groups. This is as follows: assume the BT groups G,G' are such that G' is a specialization of G. Let $\lambda_1, \ldots, \lambda_h$ (h = "height") be the slopes of G, and $\lambda'_1, \ldots, \lambda'_h$ the ones for G'. Then we have the equality

$$\sum \lambda_i = \sum \lambda'_i \qquad (1)$$

(*) This is not proved now in complete generality, but is proved if X lifts formally to char. zero, and is certainly true in general.

and the inequalities

(2) $\lambda_1 \leq \lambda'_1$, $\lambda_1 + \lambda_2 \leq \lambda'_1 + \lambda'_2$, ... $\sum_{i=1}^{j} \lambda_j \leq \sum_{i=1}^{j} \lambda'_j$ .... .

In other words, the "Newton polygon" of G (i.e. of the polynomial $\prod_i (1+(p^{\lambda_i} T))$) lies ~~above~~ below the one of G', and they have the same end-points (0,0) and (h, N) .

I get this result through a generalized Dieudonné theory ~~over~~ for BT groups over an arbitrary base S of char. p , which allows to associate to such an object an F-cristal over S, which heuristically may be thought of as a family of F-cristals, in the sense outlined above, parametrized by S. Using this theory, the result just stated is but a particular case of the analogous statement about specialisation of arbitrary cristals. Now this latter statement is not hard to prove a all: passing to $\overset{h}{\wedge} E$ and $\overset{h}{\wedge} E'$, the equality (1) is reduced to the case of a family of rank one cristals, and to the statement that ~~the same are~~ such a family is just a twist of ~~the~~ some fixed power of the (constant) Tate cristal. And the general equality (2) is reduced, passing to $\overset{i}{\wedge} E$ and $\overset{i}{\wedge} E'$, to the first inequality $\lambda_1 \leq \lambda'_1$. Raising both E and E' to a tensor-power r.th such that $r\lambda_1$ ~~and~~ is an integer, we may assume that λ_1 is an integer, and a Tate twist allows us to assume that $\lambda_1 = 0$ so the statement boils down to the following: if the general member of the family is an effective cristal, so are all others. This is readily checked in terms of the explicit definition of "cristal over S".

The ~~new~~ wishful conjecture I have in mind now is the following: the necessary conditions (1) (2) that G' ~~can~~ be a specialization of G are also sufficient. In other words, starting with a BT group $G_0 = G'$, and taking its formal modular deformation in char. p (over a formal modular variety S of dimension dd^*, $d = \dim G_0$, $d^* = \dim G_0^*$), and the BT group G over S thus obtained, we want to know if for every sequence of ~~integers~~

rational numbers λ_i between 0 and 1, satisfying (1) and (2), these numbers occur as the sequence of slopes of a fiber of G at some point of S. This does not seem too unreasonable, in view of the fact that the set of all (λ_i) satisfying the conditions just stated is indeed finite, as is of course the set of slope-types of all possible fibers of G over S.

I should mention that the inequalities (2) were suggested to me by a beautiful conjecture of Katz, which says the following : if X is smooth and proper over a finite field k, and has in dimension i Hodge numbers $h^o = h^{o,i}$, $h^1 = h^{1,i-1}$, ... , $h^i = h^{i,o}$, and if we consider the caracteristic polynomial of the arithmetic frobenius F^a operating on some reasonable cohomology group of X (say ℓ-adic for $\ell \neq p$, or cristallin), then the Newton polygon of this polynomial should be <u>above</u> the one of the polynomial $\prod(1+p^iT)^{h_i}$, In a very heuristic and also very suggestive way, this could now be interpreted by stating (without any longer assuming k finite) that the cristallin H^i of X is a specialization of a cristal whose sequence of slopes is : 0 h^o times, 1 h^1 times, ..., i h^i times. If X lifts formally to char zero, then we can introduce also the Hodge numbers of the lifted variety, which are numbers satisfying

$$h'^o \leq h^o , \ldots , h'^i \leq h^i ,$$

and one should expect a strengthening of Katz's conjecture to hold, with the h'^j replaced by the h^j . Thus the transcendental analogy of a char.p F-cristal seems to be something like a Hodge structure or a Hodge filtration, and the sequence of slopes of such a structure should be defined as the sequence in which j enters with multiplicity $h'^j = \mathrm{rank}\, Gr^j$. (NB Katz made his conjecture only for global complete intersections, however I would not be as cautious as he !). I have some idea how Katz's conjecture with the h^i's (not the h'^i's for the time being) may be attacked by the mabhinery of cristallin cohomology, at leazt the first inequality ~~xxxxx~~ among (2); on the other hand, the formal argument involving exterior powers, outlined after (2), gives the feeling that it is really the first unequality $\lambda_1 \leq \lambda'_1$ which is essential, the other should follow once we have a good general framework.

I would very much appreciate your comments to this general nonsense, most of which is certainly quite familiar to you under a different terminology. Very sincerely yours

A Grothendieck