# Letter to Anantharaman, 11.9.1967

par

Alexander Grothendieck

Transcription by

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Dear Anantharaman,

Matsumura proved that if $X$ is proper over a field $k$, then $\underline{\mathrm{Aut}}_{X/k}$ is representable by a group scheme locally of finite type over $k$. I think I can systematize the key step of his argument in the following way. Consider a scheme $S$, and a morphism

$$\varphi : Z \longrightarrow X$$

of $S$-schemes which are proper, flat and of finite presentation. Let $Y$ be locally of finite presentation and separated over $S$, then $\varphi$ induces a homomorphism of functors $u \rightsquigarrow u\varphi$:

$$\varphi' : \underline{\mathrm{Hom}}_S(X,Y) \longrightarrow \underline{\mathrm{Hom}}_S(Z,Y).$$

Then one can define a subfunctor of $\underline{\mathrm{Hom}}_S(X,Y)$ where $\varphi'$ is "unramified" in a rather obvious sense, and this turns out to be an "open subfunctor", say $\underline{\mathrm{Hom}}_S(X,Y;\varphi)$. Now look at the induced homomorphism

$$\underline{\mathrm{Hom}}_S(X,Y;\varphi) \longrightarrow \underline{\mathrm{Hom}}_S(Z,Y).$$

Using the main result of Murre's talk, one can prove that the latter morphism is representable by unramified separated morphisms locally of finite presentation ; as a consequence, if $\underline{\mathrm{Hom}}_S(Z,Y)$ is representable, so is $\underline{\mathrm{Hom}}_S(X,Y;\varphi)$.

To get, given $X$ and $Y$, a representability theorem for $\underline{\mathrm{Hom}}_S(X,Y)$, one tries to find morphisms $\varphi_i : Z_i \longrightarrow X$ as above, such that the open subfunctors $\underline{\mathrm{Hom}}_S(X,Y;\varphi_i)$ cover $\underline{\mathrm{Hom}}_S(X,Y)$ (as a fpqc sheaf), and such that the functors $\underline{\mathrm{Hom}}_S(X_i,Y)$ are all representable. If for instance $S$ is the spectrum of a field $k$, and if $X$ has "enough" points radicial over $k$ (which is always true if $k$ is alg. closed) then we can take for $Z_i$ all finite subschemes of $X$ whose points are radicial over $k$, and we get that $\underline{\mathrm{Hom}}_S(X,Y)$ i representable (any $Y$ locally of finite presentation and separated over $k$); if we do not make any assumptions on $X$ except properness over $k$, the previous assumption becomes true after finite ground-field extension $k'/k$, so that we get that for every $Y$ as above, $\underline{\mathrm{Hom}}_S(X,Y) \times_S \mathrm{Spec}(k')$ is representable. From this Matsumaras theorem stated at the beginning follows in a standard way by descent arguments. The result holds too for $\underline{\mathrm{Isom}}_k(X,Y)$ instead of $\underline{\mathrm{Aut}}_k(X)$, but as you probably know, $\underline{\mathrm{Hom}}_S(X,Y)$ is not always representable, even if $X$ is a quadratic extension of $S = \mathrm{Spec}\, k$, $Y$ being proper non projective.

Over an arbitrary base $S$, one can give a fairly general statement of a representability theorem, the points radicial over $k$ used above being replaced by suitable flat subschemes of $X$. As particular cases, we get for instance that if $X$ has integral geometric fibers and a section along which $X$ is smooth, then $\underline{\mathrm{Hom}}_S(X,Y)$ is representable; and if $X$ has reduced geometric fibers, then $\underline{\mathrm{Hom}}_S(X,Y)$ is representable locally for the étale topology over $S$. Also, if $Y$ is quasi-projective over $S = \mathrm{Spec}\, k$, then $\underline{\mathrm{Hom}}_S(X,Y)$ is representable.

To fix the ideas, I gave the statements for $\underline{\mathrm{Hom}}_S(X,Y)$, but one has quite analogous results of course for the $\prod_{X/S} P/X$ functors, which I guess will imply rather formally the other ones.

If you are interested, I can send you a photocopy of the statement of the general theorem of representability I alluded to above, and a couple of corollaries (I already listed here the most striking ones).

Sincerely yours