# Letter to J. Murre, 1962 (?)

par

Alexander Grothendieck

Transcription by

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Dear Murre,

I am glad to hear that your are still willing to give the talk on unramified functors. Here what I can say to your questions.

1. The theorem about passage to quotient I alluded to is the following:

Theorem. — *Let $f : X \longrightarrow Y$ be a morphism of $S$-preschemes, assume either $X$ and $Y$ locally of finite presentation over $S$, or $Y$ loc noeth and $X$ locally of finite type over $Y$. Assume that the equivalence relation $R = X \times_Y X$ defined by $f$ is flat over $X$ i.e. $\mathrm{pr}_1 : X \times_Y X \longrightarrow X$ is flat. Then the quotient $X/R$ exists in the strongest reasonable sense, i.e. one can factor $f$ into a compositum $X \longrightarrow Z \longrightarrow Y$, with $X \longrightarrow Z$ faithfully flat locally of finite presentation, $Z$ locally of finite pres. over $X$ (in fact of finite presentation over $S$ if $X$ is so) and $Z \longrightarrow Y$ a monomorphism.*

Of course the factorization is unique, and the theorem can be expressed by saying that the quotient sheaf (for the fpqc topology) $X/R$ is representable. That is in fact how the theorem is proved.

Raynaud has recently made a very nice (and non trivial) application of this theorem, by proving the following: if $S$ is the spectrum of a discrete valuation ring, $G$ a group prescheme of finite type over $S$, $H$ a closed and flat sub-group scheme, such that $G_t/H_t$ is quasi-affine (where $t$ is the generic point of $S$) then $G/H$ is representable as a quasi-affine and flat $S$-scheme, which is even affine if $H$ is invariant (i.e. if $G$ is a flat group scheme of finite type with affine generic fibre, than $G$ is affine). This extends immediately to a base which is regular of dim one. Raynaud is now trying to extend his construction to the case when he drops the quasi-affinness assumption, namely to construct still $G/H$ as a quasi-projective scheme over $S$.

2. Theorem of the cube.

I believe we discussed about it time ago, but maybe the proof I told you was valid only if one assumes the Pic functor of one of the factors involved representable. To prove unramifiedness of the functor <u>Corr</u> however you need only a weak infinitesimal form of the theorem of the square, for which you will find a proof in the manuscript notes I am joining on correspondence classes, containing also the proof of the statements you were recalling in your question 4. I hope you will be able to read them, I agree the handwriting is wretched and the notes moreover very sketchy. - On the other hand, I recall you that the theorem of the cube follows rather formally once one knows separatedness of $\underline{\mathrm{Corr}}_S(X,Y)$ for two of the three factors involved, and using the usual formal properties of the Picard functor (among which commutation with inverse limits of Artin rings is the less trivial).

3. As for the separatedness of $\underline{\mathrm{Corr}}_S(X,Y)$, this is about trivial whenever the Pic functor of one of the factors $X$, $Y$ is separated? Now this is certainly the case if for $X$ if its geometric fibers are integral, (a fortiori if $X$ is an abelian scheme over $S$!).

To show this, one may assume $S$ the spectrum of a discrete valuation ring, and one is reduced to show that if $\underline{L}$ is an invertible sheaf on $X$ whose restriction to the general fiber $X_t$ is trivial, then $\underline{L}$ is trivial. Now $X_1$ is an open subset of $X$, and the assumption on $\underline{L}$ can be expressed by saying that $\underline{L}$ is defined by a Cartier divisor whose support is contained in the special fiber $X_0$. Now $X_0$ itself is already a Cartier divisor (defined by a global equation $f=0$) and moreover is an integral subscheme of $X$, from this follows that the divisor $D$ is a multiple of $X_0$ (assume for simplicity the fibers of $X$ geometrically normal, and hence $X$ normal!), hence $D$ is linearly equivalent to $0$, what we wanted to prove.

I am convinced however that $\underline{\mathrm{Corr}}$ is always separated (with the usual assumptions of properness, flatness, and direct image of the structure sheaf, for both functors, of course). This is easily seen to be true if the Pic functor of either factor is representable, by a simple use of dimension theory (namely, we have a morphism $X \longrightarrow \underline{\mathrm{Pic}}_{Y/S}$ whose image has a general fiber of dimension zero, hence the same holds for the special fiber...). But it is true also, by an immediate adaptation of the same argument, if we suppose only that $\underline{\mathrm{Pic}}_{Y/S}$ is pre-ét-représentable say, i.e. is a quotient of a representable functor $Q$ by an étale equivalence relation (in fact, quasi-finite and flat would do as well), with $Q$ locally of finite type over $S$. Now this assumption is certainly satisfied if $Y$ is *projective* over $S$, as one sees by using the representation of $\underline{\mathrm{Pic}}_{Y/S}$ (or rather big open pieces of it) as the quotient of a suitable scheme of immersions of $Y$ into some $p^r$, by the action of the projective group operating freely, and taking a quasi-section of the corresponding equivalence relation…On the other hand, if one does not assume $X$ not $Y$ projective over $S$, one my think of using Chow's lemma; as $S$ is the spectrum of a discrete valuation ring, one does not loose flatness in using Chow's lemma, unfortunately one will loose however, I am afraid, the assumption $\mathrm{H}^0(X_0, \underline{O}_{X_0}) \simeq k(s)$, and I am afraid that this will make serious technical trouble. Another interesting approach, via topology, is to try to prove that under the usual assumptions on $X$, the "specialization morphism" from the fundamental group of the general geometric fiber to the one of the special fiber has an image of finite index - or at least that this is so after making the groups abelian. It seems to me that the latter statement can be proved via the Picard functor, when $X$ is assumed projective over $S$.

I am sending you some notes, including a sketch of the proof of the theorem of representability of unramified functors, although I do not think they latter can be of any use to you, as I have a hard time myself to read them. I think the notes you took when we discussed the matter a few months ago should be much more detailed; anyhow, there are certainly no simplifications in my notes relative to yours, the inverse is more plausible.

Sincerely yours