Letter to L. Bers, 15.4.1984

par

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Les Aumettes 15.4.1984

Dear Lipman Bers,

Together with Yves Ladegaillerie (a former student of mine) we are running a microseminar on the Teichmüller spaces and groups, my own motivations coming mainly from algebraic geometry, and Ladegaillerie's from his interest in the topology of surfaces. Lately we have met with a problem which I would like to submit to you, as I understand you are the main expert on Thurston's hyperbolic geometry approach to Teichmüller space. Before stating the specific problem on hyperbolic "pants" (which things boil down to), let me tell you what we are really after.

Assuming given a compact oriented surface with boundary X_0 as a reference-surface for constructing the Teichmüller-type spaces, of genus g and with "holes" (satisfying $2g-2+\nu > 0$), my primary interest is in the more "algebraic" version of Teichmüller space, corresponding to the question of classifying algebraic non singular curves over $\underline{C}$, of genus g, with a system of points (all distinct) given on X, together with a "Teichmüller rigidification" of (X, S) namely a homotopy equivalence between X_0 and $X \backslash S$. I'll denote this space, homeomorphic to $\underline{C}^d$ (where $d = 3g - 3 + \nu$), by $\widetilde{M}_{g,\nu}$ (the tilde suggesting that it is the universal covering of a finer object I am still more interested in, namely the algebraic variety (or rather "multiplicity", or "stack" in the terminology of Mumford-Deligne) of moduli for algebraic curves of type (g, ν). Thurston however considers a different modular space, where algebraic curves with a given system of points are replaced by compact conformal oriented surfaces *with boundary*, giving rise to a modular space $\widetilde{MB}_{g,\nu}$ (where the letter B recalls that we are classifying structures with boundary) homeomorphic to $\underline{C}^d \times (\underline{R}^{*+})^\nu$, where the extra factor corresponds to the extra parameters introducing through the existence of the boundary, namely the length's of the components of the boundary with respect to the canonical hyperbolic structure on the given surface. Our interest is in pinpointing the precise relationships between the two modular spaces. The obvious idea here is to consider the case of an algebraic curve with ν points given as a limit-case of a compact conformal surface with boundary, when all the lengths l_i of the components of the boundary tend to zero. Therefore, it looks suitable to consider both modular spaces above as embedded in a larger third one, which corresponds to the same modular problem as in Thurston's theory, except that we allow the "boundary" to have some components reduced to just one point, in the neighbourhood of which X is just a conformal surface without boundary, but with a given point (viewed as a component of such a "generalized boundary"). We now should get a modular space for "compact conformal oriented surfaces with generalized boundary" (of type g, ν and rigidified via X_0), call it $\widetilde{MB}_{g,\nu}$, homeomorphic to $\underline{C}^d \times (\underline{R})^\nu$, where now the second factor corresponds to the "parameters" l_i, which are allowed to take also value 0 (which means that the corresponding component of the generalized boundary is just one point). Thus $\widetilde{MB}_{g,\nu}$ appears as a variety with boundary (in the topological sense - in the real analytic sense, the "boundary" admits "corner-like" points obviously), and $\widetilde{M}_{g,\nu}$ appears as a part of the boundary.

My interest is in a better geometric understanding of the situation, which should be "intrinsic" namely not depend on any particular choice of a surgical decomposition of the reference surface X_0 into "pants", used in order to describe in a handy way standard "coordinate functions" on the modular space $\widetilde{M}_{g,\nu}$. There appears to be a geometrically meaningful retraction of $\widetilde{MB}_{g,\nu}$ upon $\widetilde{M}_{g,\nu}$ (commuting to the operations of the Teichmüller modular group), the fibers being homeomorphic to $(\mathbf{R}^+)^\nu$ - more specifically, I expect the semi-group $(\mathbf{R}^+)^I$ (where I is the set of indices for the "holes" of X_0) to act on MB in a natural way, with free action of the subgroup $(\mathbf{R}^+)^\nu$ upon $\widetilde{MB}^\circ$, in such a way that $\widetilde{M}$ is just the quotient of $\widetilde{MB}$ by this action (or of $\widetilde{MB}^\circ$ by the action of the corresponding subgroup), and that the fiber F is isomorphic to $(\mathbf{R}^+)^I$ by the choice of any "origin" in $F \cap \widetilde{MB}^\circ$.

Of course, "computationally", in terms of a decomposition of X_0 into pants, the idea of such an operation is pretty obvious - namely letting the components λ_i of $\lambda \in (\mathbb{R}^+)^I$ act as a "multiplier" on the corresponding coordinate λ_i. However, it is not clear that this operation is intrinsic - and if it were intrinsic, an intrinsic geometric description would still be desired.

Of course, in the description of the situation proposed above, the retraction of $\widetilde{MB}$ upon $\widetilde{M}$ is obtained by multiplying with the 0 multiplier (all λ are 0). Now there is a direct geometrical description of a retraction, by hyperbolic surgery. Namely, for any compact conformal surface of type g, ν with generalized boundary, let's "fill in" the holes which correspond to ordinary components of the boundary, which are Riemanian oriented circles, by "gluing in" the cones on these circles (which are canonically endowed with a conformal structure, using the Riemanian structure on the given circles). Thus we get a "functor" from compact conformal surfaces with generalized boundary (of type g, ν) to compact conformal surfaces *without boundary*, endowed with a system of ν points (making up a "wholly degenerate" generalized boundary). When we throw in the rigidifications and go over to isomorphism classes, this should give the desired retraction. However, the geometric situation is a lot richer still, as the compact surface without boundary obtained through surgery is endowed, not only with a system of ν points, but moreover with a system of mutually disjoint *discs* around these points. The shape of these discs is by o means arbitrary - we'll say that a system of discs around ν points on a compact conformal surface $\hat{X}$ without boundary is "admissible", if the situation can be obtained as above (up to isomorphism) from surgery, starting with a compact conformal surface X with boundary. (NB Among the given "discs", we should allow that some should be reduced to their center - we'll call them "degenerate".) The condition of admissibility can be expressed intrinsically, by stating that for every non-degenerate component Γ_i of the system of boundaries of those discs, the two operations we got of the standard circle group (of complex numbers of module 1) upon Γ_i, by using the fact that it is (on the one hand) the boundary of the disc D_i, an (on the other hand) that it is a component of the boundary of the hyperbolic surface $\hat{X} \backslash (\bigcup_j D_j^\circ)$, should be the same. When $\hat{X}$ and the points s_i on $\hat{X}$ are given, the possible admissible systems of discs around the points s_i depend on ν parameters - and the first idea which flips to mind to give a more precise meaning to these "parameters", is to view them as being the "radii" of those discs. But then we'll have to define what we mean by these!

The idea here is that, when we have a conformal disc D and an interior point s of D, then D may be viewed as canonically embedded in the tangent space T_s to D at s, as the "unit disc" at s. Thus, in the situation above of admissible system of discs $(D_i)_{i \in I}$ around $(s_i)_{i \in I}$, for every s_i corresponding to a non-degenerate D_i, we get a canonical disc

$$\Delta_i \subset T_{s_i}$$

in the tangent space - and of course, for degenerate D_i, we'll take Δ_i to be degenerate too. The discs we get in a given T_{s_i} (for a fixed system (s_i), and a variable admissible system of discs around these s_i) are ll discs in the strict euclidean sense, given by an unequality

$$|z| \leq r_i,$$

where $z \mapsto |z|$ denotes some hermitian metric on T_{s_i} compatible with the conformal structure - this metric being unique s_i up to a scalar factor. The set R_i of all those possible discs (the non-degenerate ones say) may be viewed in a natural way as a "torsor" (= principal homogeneous space) under $\mathbf{R}^+$, which plays here the role of the parameter space of all possible (non degenerate) "raddi" at s_i. If we admit also radius zero, we accordingly get a parameter space $\hat{R}_i$, which may be viewed as a torsor of sorts $\mathbf{R}^+$. Thus the set of radii for a given admissible system of discs D_i around the points s_i may be viewed as a point of the product-space

$$r = (r_i)_{i \in I} \in \hat{R} = \prod_{i \in I} \hat{R}_i.$$

My expectation is that an admissible set of discs (D_i) is well determined by the knowledge of the corresponding set r of radii, and moreover that a given set r of radii corresponds to an admissible system of discs iff it satisfies a set of unequalities

$$r_i < \rho_i,$$

where

$$\rho = (\rho_i)_{i \in I} \in R = \prod_{i \in I} R_i$$

is come fixed system of radii, corresponding to a fixed system of choices of hermitian metrics in the tangent spaces T_{s_i}.

I now see that this "expectation" doesn't quite match with the previous one, about a "natural operation" of $(\underline{P^+})^I$ upon $\widetilde{MB}$, having certain properties - it would match only if all ρ_i where equal to $+\infty$ (hence not in R_i itself strictly speaking). I must confess I didn't look too thoroughly yet at the situation, and moreover I've been busy with rather different kind of things for the last two or three months, and lost contact a little…

What is clear however is that the main key to an understanding of the general situation, is in an understanding of the basic particular case of Thurston's pants. If we number 0, 1 ∞ the three "holes" of such a part, the surface $\hat{X}$ can be identified canonically to the Riemann sphere, and the basic question then is to understand how the pant is embedded in this sphere Σ, as a complement of the union of (open) discs around the points 0, 1, ∞, these discs forming an "admissible system". So the main question is about understanding the structure of all possible admissible systems of three discs on Σ.

Puzzling a little about this problem, the following model came to my mind (corresponding to "limiting radii" ρ_i which are *finite*, not infinite). I view Σ as endowed with its usual euclidean metric, for which the real projective line is a great circle, with 0, 1, ∞ at equal distance from each other on this equator. These points may be viewed as the centers of three "orange slices", making up a cellular subdivision of Σ, where the common boundary of two among the "slices" Q_i ($i \in \{0, 1, \infty\}$) is a half-great circle passing in between s_i and s_j at equal distance from both, these three half-circles joining at the two poles P^+ and P^-. The "disc" Q_i around s_i has a conical structure around s_i (as has any conformal pointed disc), and we may take the concentric discs $\lambda_i Q_i$ with

$$0 < \lambda_i < 1.$$

The model I had in mind was that the (non degenerate) admissible systems of discs around the points s_i ($i \in \{0, 1, \infty\}$) are exactly the systems of discs $\lambda_i Q_i$, with λ_i as above. (If we allow some discs to be degenerate, this means that instead of the unequality above we merely demand $0 \leq \lambda_i < 1$, 0 not excluded.)

This model, if correct, would give a rather precise description of the inclusion relationships between pants, when these are considered as embedded in the sphere. The intersection of all would be this system of these half circles C_i, and the two poles P^+, P^- would play a significant role in the geometry of the pants, from this point of view. But it doesn't seem that neither those half circles (which need not be geodesical I guess), nor the two poles have ever been described as intrinsically associated to a pant. Of course, this model would give alternative "parameters" λ_i for describing a pant, which are best suited for grasping the pants in terms of spherical geometry. The next question would be an understanding of the relationship between these parameters, and Thurston's ℓ_i. Maybe it is unreasonable to expect that for given index $i \in \{0, 1, \infty\}$, the length ℓ_i depends only on λ_i an not on the other parameters λ_j - and for this reason, the intuition at the beginning of this letter, using Thurston's coordinate functions and notably the ℓ_i's to get a fibration structure on $\widetilde{MB}$ over $\widetilde{M}$, in terms of a given decomposition of X_0 into pants, is probably not really relevant, namely it is non intrinsic. Assuming the model I am suggesting is correct, the accurate description of $\widetilde{MB}$ in terms of $\widetilde{M}$ would be

$$\widetilde{MB} \simeq \widetilde{M} \times [0,1[^I,$$

where the second factor on the right hand sight refers to the system of multipliers λ_i ($i \in I$), tied to the r_i above by $r_i = \lambda_i \rho_i$.

My question of course is whether you have any information or idea to propose, especially on the basic problem of relying pants to spherical geometry, and more specifically, whether the model above is likely to hold, or is definitely false. Also, one difficulty we found with hyperbolic geometry of conformal surfaces, is that apart from existence and unicity of the hyperbolic structure (compatible with the given conformal cone and for which the boundary is geodesic), there seems to be little hold on more specific properties. As an example, starting with a compact conformal surface with boundary X (a pant, say), of hyperbolic type, and removing an (open) "collar" around the boundary, we get another surface with boundary X' - what about the relation between the two corresponding metrics? Assuming the model for pants above is correct, it would be nice to have an explicit expression of the metric of a pant in terms of the parameters λ_i.

With my thanks for your attention, and for whatever comment you will care to make, very sincerely yours