Lecture Grothendieck

Spring 70 (I think)

(Klassa)

4:10

Sheaves Godement (Hermann).

~~Affine Scheme~~

Sheaves (of sets) ① X F

(presheaf) ② $F : \mathcal{U} \longrightarrow F(\mathcal{U})$.

$U \subset V$ $F(V) \xrightarrow[\text{res}]{} F(U)$

(contravariant functor.)

③ (usual conditions)

eg sheaf of functions

holo.

diff.

etc.

Similarly define sheaf of groups, rings. ③ [~~the~~ presheaf, viewed as presheaf of sets is a sheaf]

Another way of viewing (3)

$Sh(X)$ ~~cat~~ of sheaves of sets on X.

products
inverse limits etc

Group object in $Sh(X)$ is sheaf of groups.

Ring object in $Sh(X)$ is sheaf of rings.

$\boxed{\text{sheaves of sets}}$

$$\begin{array}{ccc} G & & f_*(G) \\ X & \xrightarrow{f} & Y \\ f^*(F) & & F \end{array}$$

U open in Y

f_* defined by

~~$f_*(U) = f^{-1}(U)$~~ G

$$f_*(G)(U) = G(f^{-1}(U))$$

is sheaf.

direct image.

have functor $Sh(X) \xrightarrow{f_*} Sh(Y)$

f^* is left adjoint (unique up to iso)

$$\mathrm{Hom}(F, f_* G) \simeq \mathrm{Hom}(f^* F, G)$$

(iso of functors for fixed F, variable G).

(or fix G, F variable)

$(X, \mathcal{O}_X)$ ringed space. (X top. space, $\mathcal{O}_X$ sheaf of rings)

eg $\mathcal{O}_X(U)$ $\cancel{=}$ real valued functions on U (or values in any ring).

– if have top. ring (reals) can consider just continuous functions

– real valued differentiable functions
analytic functions.

(X, F) $\quad F_x = \varinjlim_{U \ni x} F(U)$

$x \in X$

$F \longmapsto F_x$

exact
commute with $\varinjlim$ $\varprojlim$) good functor.

$F \longmapsto G$ iso

$\Leftrightarrow F_x \longrightarrow G_x$ iso $\forall x$.

$\underline{O}_{X,x}$ in above examples of ringed spaces stalks are <u>local rings</u>.

(max ideal = germs whose value at x is zero)

$$\underline{O}_{X,x} / m_x \cong R$$

(with real valued <u>cont.</u> functions)

If just have functions — will likely not get local ring!

"sheaf of local rings". "local ringed space"

<u>Morphism between ringed spaces</u>

$$(X, \underline{O}_X) \xrightarrow{F} (Y, \underline{O}_Y)$$

(a) $X \xrightarrow{F} Y$ <u>continuous map</u>

(b) $\underline{O}_X \longleftarrow \underline{O}_Y$ <u>morphism compatible with F</u>.

$$\| $$

$$F_*(\underline{O}_X) \longleftarrow \underline{O}_Y \quad \text{(morphism of sheaves on } Y)$$

$$\| $$

$$F^*(\underline{O}_Y) \longrightarrow \underline{O}_X \quad \text{(morphism of sheaves on } X)$$

For all $\mathcal{U} \subset Y$ have map of rings

$$\mathcal{O}_Y(\mathcal{U}) \longrightarrow \mathcal{O}_X(f^{-1}(\mathcal{U}))$$

compatible with restriction.

<u>With locally ringed spaces</u> assume also that induced map on stalks is <u>local homomorphism</u>.

~~(maps max ideal → max ideal)~~

<u>ie</u> $\varphi^{-1}(m_x) = m_y$

$\Updownarrow$

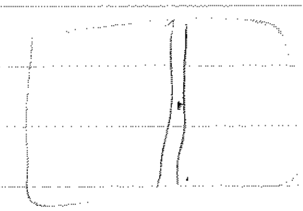
$\varphi(m_y) \subset m_x$

$$(X, F) \xrightarrow{f} (Y, G)$$

$x \in X \qquad f(x) \in Y$

$\exists$ map $G_{f(x)} \xrightarrow{\varphi} F_x$ induced by f, using $\varinjlim$

$m_y \to m_x$



<u>Locally ringed spaces form category</u>

Schemes is full subcategory of category of locally ringed spaces.

<u>For scheme</u>

Local pieces will be affine schemes:

$$A \text{ comm. ring} \longmapsto \operatorname{Spec}(A)$$

$$(X, \mathcal{O}_X)$$

locally ringed space.

$$\text{Rings} \xmapsto[\text{contravariant functor}]{} \text{locally ringed spaces.}$$

$\boxed{\text{fully faithful}}$

As set $\operatorname{Spec}(A) = \{ \mathfrak{p} \text{ prime ideal of } A \}$. $\boxed{\mathfrak{p} \neq A}$

Zariski topology

$$\forall f \in A \qquad A_f$$

$$A \longrightarrow A_f$$

$$\operatorname{Spec}(A) \xleftarrow[\text{injective}]{} \operatorname{Spec}(A_f)$$

homeo. onto X_f

value of f at $\mathfrak{p}$ is image of f in $A/\mathfrak{p}$ $= k(\mathfrak{p})$ (or its quotient field)

$$X_f = \operatorname{Spec}(A_f)$$

X_f form basis for topology.

$$\begin{tikzcd} X_f \arrow[r, mapsto] \arrow[d, phantom, "\parallel"] & A_f \arrow[d, "\wr"] \\ X_{f'} \arrow[r, mapsto] & A_{f'} \end{tikzcd}$$

have canonical isomorphism

$$\begin{tikzcd} X_f \arrow[r, mapsto] \arrow[d, hook] & A_f \\ X_{f'} \arrow[r, mapsto] & A_{f'} \arrow[u, "F"] \end{tikzcd}$$

F unique map making commutative

$$\begin{tikzcd} A_{f'} \arrow[r] & A_f \\ & A \arrow[ul] \arrow[u] \end{tikzcd}$$

have "presheaf" on basis.

has properties of sheaf on basis.

~~can always extend to sheaf on all open sets.~~ EGA I Mumford.

- also is sheaf of local rings

$U = \bigcup X_{f_i}$

$$\mathcal{O}_X(U) \cong \operatorname{Ker}\left[\left(\prod \mathcal{O}_X(U_{f_i})\right) \rightrightarrows \prod_{i,j} \mathcal{O}_X(U_{f_i} \cap U_{g_j})\right]$$

A_f is solution of universal problem:

$g(f)$ invertible.

$$\begin{tikzcd} A \arrow[d] \arrow[dr, "g"] & \\ A_f \arrow[r, dashed, "\exists h"'] & B \end{tikzcd}$$

$\mathcal{O}_{X,x} = \cancel{A_{\mathfrak{p}}}\ A_{\mathfrak{p}}.$ local ring

$\boxed{\text{Krull's thm } f \in A^{\times} \iff \text{"value" of } f \text{ is never zero}}$

$x = \mathfrak{p}$ prim

q.f. of $A/\mathfrak{p} = A_{\mathfrak{p}}/\mathfrak{m}_{\mathfrak{p}}$

$$\begin{array}{ccccccc}
X_f & A & \longmapsto & \operatorname{Spec} A & \varphi^{-1}(\mathfrak{p}) & = X & \mathcal{O}_X \\
& \downarrow \varphi & & \uparrow & & & \downarrow \\
& B & & \operatorname{Spec} B & \mathfrak{p} & = Y & \mathcal{O}_Y
\end{array}$$

need define only on basis.

$$\varphi^{-1}(X_f) = Y_{\varphi(f)}$$

$$\begin{array}{ccc}
A & \longrightarrow & B \\
\downarrow & & \downarrow \\
A_f & \longrightarrow & B_{\varphi(f)}.
\end{array}$$

have local homomorphism of sheaves of rings.

$$A_{\varphi^{-1}(\mathfrak{p})} \longrightarrow B_{\mathfrak{p}}.$$

Rings $\longmapsto$ Locally ringed spaces.

(is fully faithful).

$$A \longmapsto (X, \mathcal{O}_X) \longrightarrow \mathcal{O}_X(X) = A$$

Groupe Schemes
Demazure + Gabriel

Jan 15 Office Hours
2nd lecture
Wednesday } after 2 in Ribenboim's office.
Friday }

Formal schemes similar ideas.

Affine scheme — ringed space isomorphic to

$$A \xrightarrow[\text{[fully faithful]}]{} \operatorname{Spec} A.$$

$$(X, \mathcal{O}_X)$$

φ local morphism

$S = \operatorname{Spec} A$ induces $\Gamma\varphi : A \longrightarrow \Gamma(X, \mathcal{O}_X)$

(bijection) $A = \Gamma(S)$

ie $\mathrm{Hom}_{\text{lo. ring space}}(X, S) \xrightarrow{\sim} \mathrm{Hom}_{\text{ring}}(A, \Gamma(X, \mathcal{O}_X))$

$\varphi \in$

(turns X into sheaf of A-algebras)

pre) **Scheme** = ringed space $(X, \mathcal{O}_X)$

such that $X = \bigcup_i X_i$ such that

$(X_i, \mathcal{O}_X | X_i)$ is an affine scheme

(~~prescheme~~ is changed to scheme in EGA IV)

$X \xrightarrow{f} Y$

morphisms of schemes are always local

$\mathcal{O}_X \leftarrow - - \mathcal{O}_Y$

nonlocal formulation of local morphism

$f^{-1}(U)$ $\quad$ U

$\varphi' = f^{-1}(\varphi)$ $\quad$ $\varphi \in \mathcal{O}_Y(U)$ $\quad$ U_φ = largest open subset of U on which φ invertible require $f^{-1}(\mathcal{U}_\varphi) = \mathcal{U}_{f^*\varphi} \cdot f^{-1}(\mathcal{U})_{\varphi'}$ (check)

$$X_\varphi = \{x \in X \mid \varphi(x) \neq 0\} \qquad \varphi \in \Gamma(X, \mathcal{O}_X)$$

local ringed space

$\|$

max open set on which φ invertible

$\exists$

$$\begin{tikzcd} & X \times_Z Y \arrow[dl] \arrow[dr] & \\ X \arrow[dr] & & Y \arrow[dl] \\ & Z & \end{tikzcd}$$

cartesian product

$$\begin{tikzcd} X \arrow[d] \\ Z \end{tikzcd}$$

schemes over Z

morphism in

$$\begin{tikzcd} X \arrow[r] \arrow[dr] & Y \arrow[d] \\ & Z \end{tikzcd}$$

$\operatorname{Spec} \mathbb{Z}$ = final object (e) $\exists!$ morphism $\begin{tikzcd} X \arrow[d] \\ \operatorname{Spec}\mathbb{Z} = e \end{tikzcd}$ $\Rightarrow \exists$ ordinary product (fibre product over e)

[final for locally ringed spaces as well]

$\exists$ finite inverse limits (OK whenever $\exists$ fibre product).

$\&$

$$\begin{tikzcd} Z \arrow[r, "w"] & X \arrow[r, shift left, "u"] \arrow[r, shift right, "v"'] & Y \end{tikzcd}$$

kernel of pair u, v

if ① $u \circ w = v \circ w$

② universal with this property

$$\begin{tikzcd} T \arrow[d, "f"'] \arrow[dr, "\varphi"] & & \\ Z \arrow[r, "w"'] & X \arrow[r, shift left, "u"] \arrow[r, shift right, "v"'] & Y \end{tikzcd}$$

if $u \circ \varphi = v \circ \varphi$

then $\exists!\, f$.

Kernel exists.

$$\begin{tikzcd} X \arrow[r, "(u,v)"] & Y \times Y \\ Z \arrow[u] \arrow[r] & Y \arrow[u, "\Delta"'] \end{tikzcd}$$

fibre product of (u,v) and Δ

Then Z is kernel of (u,v).

<u>Proof</u> If X, Y, Z affine

$$\begin{tikzcd} X = \operatorname{Spec} B \arrow[dr] & & Y = \operatorname{Spec} C \arrow[dl] \\ & Z = \operatorname{Spec} A & \end{tikzcd} \qquad \begin{tikzcd} & B \otimes_A C & \\ B \arrow[ur] & & C \arrow[ul] \\ & A \arrow[ul] \arrow[ur] & \end{tikzcd}$$

$$X \times_Z Y = \operatorname{Spec} B \otimes_A C.$$

(is fibre product in category of locally ringed spaces!)

Check UMP (

$$\begin{tikzcd} T \arrow[dr] & \\ X \times Y \arrow[r] & Y \end{tikzcd}$$

$$\Gamma(T, \mathcal{O}_T) = D.$$

by above: $\operatorname{Hom}(T, X \times Y)$

$$= \operatorname{Hom}(B \otimes_A C) \longrightarrow D$$

(use UMP for $\otimes$)

? $\exists$ fibre product for category of locally ringed spaces. not sure.

Patch for more general case. EGA I.

$X \qquad A = \Gamma(X, \mathcal{O}_X)$.

$\exists$ natural map $X \longrightarrow \operatorname{Spec} A$.

eg $X =$ projective space. Not embedding!

A is most general affine ground scheme.

$$\begin{tikzcd} X \arrow[d, "f"'] \arrow[dr] & \\ \operatorname{Spec} B & \operatorname{Spec} A \arrow[l] \end{tikzcd}$$

f factors thru $\operatorname{Spec} A$.

$\mathcal{C}$ category.

$\hat{\mathcal{C}}$ = contravariant functors $\mathcal{C} \longrightarrow$ Sets.

$= \operatorname{Hom}(\mathcal{C}^\circ, \text{Sets})$

$\mathcal{C}$ is iso to full subcategory of $\hat{\mathcal{C}}$

$\mathcal{C} \xhookrightarrow{h} \hat{\mathcal{C}}$

$X \in \mathcal{C}$

$X(Y) = h(X)(Y)$ pts of X with values in Y.

$X \longmapsto \underbrace{\left(Y \longrightarrow \operatorname{Hom}(Y, X)\right)}_{h(X)}$

Product in $\hat{C}$ is taken element-wise (sets)

e.g. $(X \times X')(Y) = X(Y) \times X'(Y)$ product of sets (def. of product)

e.g. $(X \times_Z X')(Y) = X(Y) \times_{Z(Y)} X'(Y)$

e.g. $X \underset{v}{\overset{u}{\rightrightarrows}} X'$ $\ker(u,v)(Y) = \ker(u(Y), v(Y))$; $X(Y) \rightrightarrows X'(Y)$

Image of h = representable functor.

Sometimes get out of C — have to work in $\hat{C}$.

e.g. $C = (\text{Schemes}) \supset (\text{Affine schemes}) = (\text{Rings})^{\circ}$.

$X \longmapsto \left(X(A) = X(\operatorname{Spec} A) = \operatorname{Hom}(\operatorname{Spec} A, X) \right)$ A ring. functor known if known on affine schemes.

i.e.

$(\text{Schemes}) \xrightarrow[\text{fully faithful functor}]{} \underline{\operatorname{Hom}}(\text{rings}, \text{sets})$ = covariant functors on category of rings.

$Y \supset \mathcal{U}$ $\qquad$ $\mathcal{U} \to \mathrm{Hom}(\mathcal{U}, X)$

gives sheaf on Y.

M. Artin — models affine schemes by étale topology "algebraic?")

here can characterize which functors (rings → sets)

arise from "schemes" (of finite type over field - ring). $\boxed{\text{Deep}}$

Jan 20

pts. fix ring K $\qquad$ index set I

K-pts of "affine space" dim I over K $\Big\}$ $K^I \ni x = (x_i)_{i \in I}$ $\qquad x_i \in K$

$\cup$

X = K-alg. subset: $\qquad P = K[(X_i)_{i\in I}]$

$F_j \in P \quad j \in J$

$$X = V(\{F_j\})$$

$\ni F$

$$\| $$

$$\{x \in K^I \mid F_j x = 0 \ \forall j \in J\}. \qquad K^I \xrightarrow{F} K$$

$$\|$$

$$V(\mathcal{J}) = V(\mathcal{J})(K) \qquad \mathcal{J} = \text{ideal generated by } \{F_j\}.$$

Consider also all (commutative) K-algebras $K \to K'$.

$$\begin{tikzcd} X(J)(K) \arrow[d] & \subset & K^I \arrow[d] \\ X(J)(K') & \subset & K'^I \end{tikzcd} \quad \longleftarrow \text{"}J\text{"}$$

Two sets of equations equivalent if they give same subfunctor of $(\ )^I$ (on category of commutative K-algebras) ie same set of points in any K'^I.

$$\begin{tikzcd} X \arrow[d, phantom, "\cap"] & \Leftrightarrow_{\mathrm{def}} & X_{(K')} \arrow[d, phantom, "\cap"] & \subset & K'^I \\ Y & & Y_{(K')} & \subset & K'^I \end{tikzcd} \quad \forall K',$$

Much easier than insisting points only in K.

In fact $\begin{matrix} X & = & Y \\ \| & & \| \\ V(J) & & V(K) \end{matrix} \iff J = K$. In fact $X \subset Y \iff J \supset K$.

(Test K': P/J or P/K)

Such X, Y are affine schemes over K: here represented as subfunctors of $(\ )^I$.

$$X = V(\mathcal{J}) \qquad \underset{\text{ideal}}{\mathcal{J}} \subset P$$

$$X(K') \subset K'^{I} \cong \operatorname{Hom}_{k\text{-alg}}(P, K')$$

bij.

functorial in K'.

$$\begin{array}{ccc} V(K') & \subset & K'^{I} \\ \wr\| & & \wr\| \\ \operatorname{Hom}_{k\text{-alg}}(A, K') & \subset & \operatorname{Hom}_{k\text{-alg}}(P, K') \end{array}$$

$$A = P/\mathcal{J}$$

Start with any A, write it as $P/\mathcal{J}$.

— i.e. describes the above as a trivial answer to the question of which functors are representable.

$$\begin{array}{ccc} X & \subset & E_K^I = K^I \\ \downarrow & & \\ Y & \subset & E_K^L \end{array}$$

morphism of functors (algs $\longrightarrow$ Sets)

or: polynomial maps $E_K^I \longrightarrow E_K^L$ taking $X \longrightarrow Y$

(i.e. $X(K') \longrightarrow Y(K')$ $\forall K'$.

Now throw out the ambient affine space; use the functor antiequivalence X as above $\longleftrightarrow$ K-algebras.

$$X(K') \cong \mathrm{Hom}_{K\text{-alg}}(A, K')$$

viz To K-algebra A corresponds functor

$$\mathrm{Spec}(A) : K' \longrightarrow \mathrm{Hom}_{K\text{-alg}}(A, K')$$

$$\underset{\text{points of } X \text{ in } K'}{X(K')} = \mathrm{Hom}_{K\text{-alg}}(A, K') \underset{\text{Yoneda!}}{=} \mathrm{Hom}_{K\text{-Schemes}}(\mathrm{Spec}\,K', \underset{\mathrm{Spec}(A)}{\overset{\|}{X}})$$

Proof

$$\begin{array}{ccccccc} X & & A & = & P_I/\mathcal{J} & \longleftarrow & P_I \\ \downarrow & \underset{\text{Yon.}}{\longleftrightarrow} & \uparrow & & & & \uparrow \\ Y & & B & = & P_L/\mathcal{K} & \longleftarrow & P_L \end{array} \qquad \exists, = \{H_\ell : E_K^I \to K_K^L\}$$

$$\begin{array}{l} Y \\ \cap \\ X \subset E_K^I \\ \| \\ V(\mathcal{J}) = \mathrm{Spec}\left(\frac{P}{\mathcal{J}}\right). \end{array}$$

$Y = V(\mathcal{K}) \quad \mathcal{K} \supset \mathcal{J}$

$\longleftrightarrow \mathcal{K}$ ideal of $P/\mathcal{J}$

subschemes of $X \overset{1:1}{\longleftrightarrow}$ ideals of co-ordinate ring of X.

Everything breaks down if fields only:

$X = Y \iff \mathcal{J} = \mathcal{K}$ fails — and is replaced by $X = Y \iff \sqrt{\mathcal{J}} = \sqrt{\mathcal{K}}$.

Same if restrict to K, K' reduced: no nilpotents.

§3 The two approaches united

X scheme

cat. functors

$$(Sch) \longrightarrow \mathrm{Hom}(\text{Rings}, \text{Sets})$$

$$X \longmapsto (A \mapsto \mathrm{Hom}(\mathrm{Spec}\,A, X))$$

points of X in A.

If ~~of~~ we know the points in all A, how to recover X?

Parenthetical k field. Determine all

$$\xi = \mathrm{Spec}(k) \longrightarrow X$$

is just X with

$$\begin{matrix} X \\ \Psi \\ x \end{matrix}$$

~~with~~

with $\underline{O}_{X,x} \longrightarrow k$ local.

$$\begin{tikzcd} \underline{O}_{X,x} \arrow[r] \arrow[d] & k \\ \underline{O}_{X,x}/m_x \arrow[ur, dashed] & \end{tikzcd}$$

$m_x \subset \underline{O}_{X,x}$

$k(x) = \underline{O}_{X,x}/m_x$

so is just $x \in X$ with $k(x) \hookrightarrow k$. It is a k-point of X.

Now suppose $\xi = \operatorname{Spec} k \to X$, $\xi' = \operatorname{Spec} k' \to X$

ie $x, x' \in X$ $\quad k(x) \hookrightarrow k \quad k(x') \hookrightarrow k'$.

$$x = x' \iff \exists \; \xi^* = \operatorname{Spec} k^* \quad \begin{tikzcd} & \operatorname{Spec} k' \arrow[dr] & \\ \operatorname{Spec} k^* \arrow[ur] \arrow[dr] & & X \\ & \operatorname{Spec} k \arrow[ur] & \end{tikzcd}$$

Every point of ~~the~~ X is image of a k-valued point, for some k ~~and~~ $x = x' \iff$ above diagram ~~true~~.

As a set, X = quotient of $\bigcup X(k)$ $\forall$ fields k

(for some k, $X(k) = \phi$).

[illegible]

Topologically
$$\begin{array}{c} \mathcal{U} = \bigcup_i X_{f_i} \\ \cap \\ X = \operatorname{Spec}(A) \end{array}$$

~~[illegible]~~

see [illegible]

as functor

$$\mathcal{U}(\ ) \subset X(\ )$$

$$\mathcal{U}(K) \subset X(K) \qquad K \text{ commutative ring}$$

$$\begin{array}{c} \psi \\ \varphi : A \to K \end{array}$$

$$\begin{tikzcd} & \mathcal{U} \arrow[d, phantom, "\cap"] \\ Y = \operatorname{Spec} K \arrow[ur, dashed, "?"] \arrow[r, "\varphi"] & \operatorname{Spec} A \end{tikzcd}$$

~~[illegible]~~ ?

Y maps into $\mathcal{U}$

$\Rightarrow Y = \bigcup Y_{\varphi(f_i)} \iff (\varphi(f_i)) =$ unit ideal

The open subfunctors are those for which

$$\left.\begin{array}{c} \varphi : A \longrightarrow K \\ \psi \\ \exists\, f_i \qquad (\varphi(f_i)) = (1) \end{array}\right) \forall K \text{ simultaneously}$$

Definition $\quad X$ any functor (rings $\to$ sets)

$$\begin{array}{c} \cup \\ \mathcal{U} \end{array} \text{ a subfunctor } \quad (\text{i.e. } \mathcal{U}(S) \subset X(S))$$

I don't understand
this difference ...
in Demazure
Gabriel

$\mathcal{U}$ is an open subfunctor $\Leftrightarrow$ it is

(a) a sheaf in the Zariski topology

$$S = \operatorname{Spec}(K) \longrightarrow U(S)$$

K comm ring

S_g restriction to S_g has properties of sheaf.

$g \in K$

([illegible])

i.e. if $S_g = \bigcup S_{g_i}$

$$U(S_g) \longrightarrow \prod_i U(S_{g_i}) \rightrightarrows \prod_{i,j} U(S_{g_i g_j})$$

Any functor represented by a scheme has this property.

OK.

(b) given

$$\begin{array}{c} X \\ \uparrow{\scriptstyle \varphi} \\ \operatorname{Spec} K = S \end{array} \qquad \varphi \in X(K)$$

construct fibre product

$$\begin{array}{ccc} X & \overset{i}{\supset} & \mathcal{U} \\ \uparrow{\scriptstyle \varphi} & & \uparrow{\scriptstyle \text{"}\varphi\text{"}} \\ S & \longleftarrow & S_{\mathcal{U}} \end{array}$$

$S_{\mathcal{U}}$ is open in previous sense.

$S_{\mathcal{U}} =$ "φ^{-1}" Cartesian square in $\hat{\mathcal{E}}$, (arguments). $\boxed{\text{can check in case with scheme}}$

If above X is a scheme — there are

precisely the open sets.

$$X \supset \mathcal{U} \text{ open.}$$

$$X(k) \supset \mathcal{U}(k).$$

What is a sheaf?

$$X \supset \mathcal{U} \quad \text{open subfunctor}$$

$$\underset{X}{\mathcal{O}}(\mathcal{U}) = ?$$

$$\mathcal{O} = \operatorname{Spec} \mathbb{Z}[t] = E^1_{\mathbb{Z}} \qquad \underline{\mathcal{O}}_X(\mathcal{U}) = \operatorname{Hom}(\mathcal{U}, \mathcal{O}).$$

<u>X is a scheme</u> $\Leftrightarrow$

(a) is sheaf for Zariski topology

(b) $X \supset X_i$ open subfunctors such

that (1) X_i is affine scheme — i.e. representable by rings

(2) $\forall$ field K $\quad X(K) = \bigcup X_i(K)$

is representable functors, glued together nicely.

From now on, everything is affine:

Fibred Products   Schemes X, S' over S

$$X' = X \times_S S' \quad \text{as inverse image, the scheme over } S'$$

this is homeomorphism onto its image   "base change".

$$\begin{tikzcd} X \arrow[d] & X_s \arrow[l] \arrow[d] \\ S & s \arrow[l] \end{tikzcd} \qquad \begin{matrix} (\text{fibre of } X \text{ at } s). \\ \\ = \mathrm{Spec}(k(s)). \end{matrix}$$

(1pt).

defined from $\underline{O}_{S,s} \longrightarrow k(s)$.

$X_s \cong$ points over s.

$$\underline{O}_{X,x} \longrightarrow \underline{O}_{X_s,x} \quad \text{surjective, ker } m_s \underline{O}_{X,x}.$$

$$\underline{O}_{S,s} \longrightarrow \underline{O}_{X,x} \longrightarrow \underline{O}_{X_s,x} \cong \frac{\underline{O}_{X,x}}{m_s \underline{O}_{X,x}}.$$

In algebraic terms

$$\begin{array}{ccccccc} B & \longrightarrow & B' = B \otimes_A A' & & B/\mathfrak{p}B & \longleftarrow & B \\ \uparrow & & & & \uparrow & & \\ A & \longrightarrow & A' = k(\mathfrak{p}) & \supset & A/\mathfrak{p} & \longrightarrow & k(\mathfrak{p}) \end{array}$$

?

$A' = k(\mathfrak{p})$: field

X, $\mathcal{O}_X$ -ringed space. $\mathcal{O}_X$-module $\mathcal{M}$.

Case $X = \operatorname{Spec} A$. Then $\mathcal{O}_X$ is sheaf of A-algebras.

$M \in \operatorname{Mod} A$. $\tilde{M} = M \otimes_A \mathcal{O}_X$ is

sheaf of modules over $\mathcal{O}_X$.

alternate description $\tilde{M}(X_f) = M_f$.

$\sim$: $\operatorname{Mod}(A) \longrightarrow \operatorname{Mod}(\mathcal{O}_X)$.

functor

fully faithful and exact, and commutes

with all $\varinjlim$ ($\Rightarrow$ w. all $\oplus$).

When is an $\mathcal{O}_X$-module $\cong$ to an $\tilde{M}$?
"ie in the "essential image" of

<u>Definition</u> $X, \mathcal{O}_X$ $\mathcal{M}$ is quasi-coherent

if $\exists$ $X = \bigcup X_i$ s.t.

$$\mathcal{M}|X_i \cong \mathrm{Coker}\left(\mathcal{O}_{X_i}^{(I)} \longrightarrow \mathcal{O}_{X_i}^{(J)}\right) \qquad I, J \text{ not necess. finite.}$$

All $\tilde{M}$'s are quasi-coherent (clearly).

Indeed the $\tilde{M}$'s are precisely the ones which are cokers of $\mathcal{O}_X^{(I)} \longrightarrow \mathcal{O}_X^{(J)}$. The slightly subtler point is that it is also equivalent to locally same property ie ess. image = quasi-coherent.

$$\underline{\text{Closed subschemes of } \operatorname{Spec} A} \iff \boxed{\text{quasi-coherent } \mathcal{O}_X\text{-ideals}}$$

$$J \subset A$$

$$\operatorname{Spec} A \longleftarrow V(J) = \operatorname{Spec}(A/J) \quad \Big\| \quad \mathcal{J} \subset \mathcal{O}_X.$$

Immersions $Y \xrightarrow{i} X$ immersion $\Leftrightarrow$

$\forall y \in Y,\quad x = i(y) \quad \exists\ X' \ni x$

$\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad \shortparallel$

$\quad\quad\quad\quad\quad\quad\quad\quad\quad\ \operatorname{Spec} A$

$$\begin{array}{ccc} i^{-1}(X') = Y' & \longrightarrow & X' \\ \wr\shortparallel & & \shortparallel \\ \operatorname{Spec} A/J & \xrightarrow[\text{canonical}]{} & \operatorname{Spec} A \end{array}$$

Y need not be affine

open immersion – if can do with $J = 0$,

X affine $\quad X_f \hookrightarrow X$ is open immersion.

Also $Y \xrightarrow[\text{inclusion}]{} X$

$\ \uparrow$
open.

Prop. (a) i is homeomorphism of Y to locally closed subset $i(Y) \subset X$.

(b) $i^{-1}(\mathcal{O}_X) \longrightarrow \mathcal{O}_Y$ epimorphism: ($*$?)

surjective. $\mathcal{O}_{Y,y} \longleftarrow \mathcal{O}_{X,x}$ $(x = i(y))$

(c) i is monomorphism in category of schemes, and even of all ringed spaces.

Counterexample

to (a), (b) ⇒ immersion:

if locally closed but not: $x \longrightarrow X$ immersion of closed point.

(Apparently a), b) as given ⇒ immersion, EGA.

e.g. — J nilpotent

$$\begin{array}{ccc} \operatorname{Spec}\left(A/J\right) & \longrightarrow & \operatorname{Spec} A \\ \| & & \| \\ Y & & X \end{array}$$

bijective ≠ ≅ homeomorphism?

immersion?

<u>Definition</u> Y, Y' define same subscheme of X if ∃ ≅ making △ commute.

$$\begin{array}{ccc} Y & \xleftarrow{\cong} & Y' \\ & {\scriptstyle i}\searrow \quad \swarrow{\scriptstyle i'} & \\ & X & \end{array}$$

(general procedure in category, from image to subobject).

<u>Definition</u> An immersion i is closed if its image is closed.

If $X = \operatorname{Spec} A$, the only such are the $\operatorname{Spec}(A/J)$ (quasi-coherent ideals of $\mathcal{O}_X$).

Indeed $\forall X$, the closed subschemes $\overset{1-1}{\longleftrightarrow}$ quasi-coherent sheaves of ideals, as follows:

$$Y \hookrightarrow X \text{ closed subschemes.}$$

$$i_*(\mathcal{O}_Y) \longleftarrow \mathcal{O}_X \longleftarrow J \;=\; \text{kernel}$$

Any immersion factors into closed immersion followed by an open immersion quite ... which smallest subscheme be ?

Y locally closed. $\bar{Y} - Y = \dot{Y}$

$Y \subset U = X - \dot{Y}$ is biggest open set which works. (= canonical choice).

Immersion + base change

$$\begin{tikzcd} Y \arrow[d, hook] & Y' \arrow[l] \arrow[d, hook] \\ X & X' \arrow[l] \end{tikzcd}$$

immersion $\Rightarrow$ immersion

cl. " $\Rightarrow$ closed "

open " $\Rightarrow$ open "

Projective Spaces

k field E finite dim'l vector space

$$P' = P'(E)(k) = \{\text{subspaces of dimension } 1\}.$$

$E = k^{n+1}$ homogeneous co-ordinates ie

(not proportionality).

$$\check{E} = \mathrm{Hom}_k(E, k)$$

$$P(E)(k) = \{\text{quotient vector spaces of } E \text{ of rank } 1\}$$

so $P(E)(k) = P'(\check{E})(k)$.

Choose basis k.

$$P(E)(k) = \bigcup_{i=0}^{n} U_i \quad (\cong k^n)$$

as follows

$E \longrightarrow L$ quotient dim 1.

$e_0, e_1, \dots e_r$ $\quad U_i = $ ~~the set ...~~ e_i does not go to 0 in L

<u>Schemify</u>

$$\begin{tikzcd}
\text{quasi-coherent } E \; (\text{e.g. } \mathcal{O}_S^{r+1}) \arrow[d, dash] & E_{S'} \arrow[d] \\
\text{scheme } S & S' \arrow[l]
\end{tikzcd}$$

functor on schemes over S.

$$P(E)(S') = \left\{ \begin{array}{l} \text{all quotient sheaves of } E_{S'} \\ \text{which are locally free of rank } 1 \end{array} \right\}$$

is a contravariant functor

<u>Exercise</u> This functor is representable, and the representing object is called $P(E)$, the projective scheme associated to E.

<u>Vector bundle functor</u> $\mathbb{V}$

quasi-coherent E $\qquad E_{S'}$

$$\begin{array}{c} E \\ \big| \\ S \longleftarrow S' \end{array}$$

$$\mathbb{V}(E)(S') = \mathrm{Hom}(E_{S'}, \mathcal{O}_{S'})$$

this is representable ~~also~~.

$$\mathrm{Sym}^*_A(M) = \coprod_i \mathrm{Sym}^i_A(M)$$

symmetric algebra.

$$\begin{tikzcd} \mathrm{Sym}(M) \arrow[dr, "\exists !"] & \\ M \arrow[u, phantom, "\cup"] \arrow[r] & A' \end{tikzcd}$$

Grothendieck 5

X scheme (can be affine).

$$P(E) : (\text{Schemes}/S)^{\circ} \longrightarrow \text{Sets}$$

is representable functor. Representing object will be bundle defined by E.

$$\begin{tikzcd} M & E \arrow[d] & \\ & & E_{S'} \arrow[ull, dash] \arrow[d] \\ A & S & S' \arrow[l, "f"] \end{tikzcd}$$

E quasi coh. mod.

$A \longrightarrow A'$

$$f^*(E) \quad \boxed{f^{-1}(E) = f^{-1}(O_S)}$$

$$f^{-1}(O_S) \longrightarrow O_{S'} .$$

$$f^*(E) =_{df} f^{-1}(E) \otimes_{f^{-1}(O_S)} O_{S'}$$

(corresponds to extension of scalars in affine case.)

$$\boxed{M' = M \otimes_A A'}$$

f^* commutes with infinite direct sums

$$f^*(O_S^{(I)}) = O_{S'}^{(I)}$$

$S' \longrightarrow$ set of all quotients of $E_{S'}$ which are locally free of rank 1.

Is functor w.r.t. S'.

$$P(E)(S') \longrightarrow P(E)(S'') \quad \text{if } S'' \longrightarrow S'.$$

(in affine case $P(E)(A')$ = set of quotients A'-module L ~~of~~ of $M \otimes_A A'$ which are projective of rank 1

M

i.e. $\exists \varphi_i \in A$, $A = \sum \varphi_i A$ s. that $L_{\varphi_i} \cong A_{\varphi_i}$ $\forall i$.)

All conditions local in Zariski topology.

To check condition to be scheme (in sense of previous lectures).

$e_i \in M$, $i \in I$, $P \supset P_i$

1) $P_i(A')$ = sets of quotients L' of $M \otimes_A A'$ such that $\varphi(e_i)$ is a basis of L',

$P_i S' = \ldots$ $\quad E'_S \xrightarrow{\varphi} L' \ldots \varphi(e_i) \ldots L'$.

1) P_i open subfunctors

2) they cover P

3) they are representable by rings

$\boxed{\text{check in turn}}$

(1) subsheaf in Z. topology). ✓

2) must show ; for in S_i ?)

$$\begin{array}{ccccc}
P_i & \subset & P & & \\
\vdots & & \uparrow & & \\
S_i & \hookrightarrow & S & L' & \xleftarrow{q} E_{S'} \\
 & & \uparrow & & \\
 & & T & L'_T & \longleftarrow E_T
\end{array}$$

(Nakayama's lemma)

(3) $E \xrightarrow{u} F$ $\quad$ u isom $\Longleftrightarrow$ $u(s) : E(s) \xrightarrow{\sim} F(s)$

$$\begin{array}{c} E(s) \\ \| \\ E_s \otimes k(s) \\ \cup \\ \xi \end{array}$$

$\cancel{J}$

$\mathcal{O}^u \longrightarrow \mathcal{O}^u.$

(u_{ij}) invertible $\Leftrightarrow$ $\det(u_{ij})$ is unit

$\Leftrightarrow$ invertible modulo maximal ideal.

so get P_i open subfunctor.

(2) They cover P. $P_i \subset P$

$S_i' \subset S'$

need S_i' cover S' etc.

enough to take S' point (spec field).

$$\begin{array}{ccc} M & \xrightarrow{\varphi} & L \\ | & & \\ K & & \end{array}$$

(3) prove each is representable by a ring.

(by $\operatorname{Spec} A[X_i]_{i \in I}$ in free case)

leave as exercise

(have proved $\Rightarrow$ E locally free)

$$\begin{tikzcd} P \arrow[r, dash] \arrow[d, dash] & P \times_S S' \arrow[d, dash] & \\ S \arrow[r, dash] & S' \arrow[r, dash] & T \end{tikzcd}$$

$$P'(T) = \mathrm{Hom}_{S'}(T, P')$$

$$= \mathrm{Hom}_S(T, P) \qquad \text{"restriction"}$$

(get grassmannian with quotients locally free of rank >1 ...)
also representable

$$\mathrm{Grass}_n(E) \hookrightarrow \mathbf{P}(\Lambda^n E)$$
$$\qquad\qquad\qquad \| $$
$$\qquad\qquad\qquad \mathrm{Grass}_1(E)$$

$$\begin{tikzcd} P(E) \arrow[r] \arrow[d, dash] & P(E)_{S'} \simeq P(E_{S'}) \arrow[d, dash] \\ S \arrow[r, dash] & S' \end{tikzcd}$$

Group Objects

$$\mathcal{C} \xhookrightarrow{h} \hat{\mathcal{C}} \quad \left(\underline{\mathrm{Hom}}(\mathcal{C}^{\circ}, \mathrm{Sets}) \right)$$

$$\hat{\mathcal{C}} \ni F$$

$$F : \mathcal{C}^{\circ} \longrightarrow \mathrm{Sets}$$

give comp on $F(X)$ $\forall X$ such that

$$\begin{array}{ccc} F(X) & & X \\ \uparrow & & \downarrow \\ F(Y) & & Y \end{array}$$

is functorial.

ie $F(X) \times F(X) \longrightarrow F(X)$

$$(F \times F)(X)$$

ie natural transformation $F \times F \longrightarrow F$

could do with more than one functor

~~Eg~~ Associative — associative on all $\mathbf{F}(X)$

etc.

Could be ring object, group object etc.
Lie ring etc.

Can also have

A, M $\in \hat{\mathcal{C}}$

ring, module over A

[rings will be associative with unit, modules unitary]

F with composition

$$F \times F \xrightarrow{\varphi} F$$

associativity means

$$\begin{array}{ccc} F \times F \times F & \xrightarrow{(\varphi \times id)} & F \times F \\ \downarrow{\scriptstyle (id) \times \varphi} & & \downarrow{\scriptstyle \varphi} \\ F \times F & \xrightarrow[\varphi]{} & F \end{array}$$

is commutative diagram.

can say $\forall X$, $\varphi(X)$ is associative.

Shorter!

$\bar{F}$ Similarly for commutative law

$$\begin{array}{ccc} F\times F & \xrightarrow{\varphi} & F \\ \downarrow s & \nearrow \varphi & \\ F\times F & & \end{array}$$

$e \in \hat{C}$ final

$e(X) = \{\phi\}$.

(has one point)

$$F \underset{\varepsilon}{\overset{\longrightarrow}{\longleftarrow}} e$$

F as pointed functor (ie values in category of pointed sets).

$F \quad \varphi, \varepsilon$

$$\begin{array}{ccc} F & \longrightarrow & e\times F \\ \| & & \downarrow \varepsilon\times id \\ F & \xleftarrow{\varphi} & F\times F \end{array}$$

diagrammatic way to say ε is left unit.

Nicer to just reduce to $F(X)$ case. Don't need to give diagrammatic proofs.

$$C = (\text{Schemes})$$

group object over S.

$$\begin{tikzcd} C \arrow[d, dash] \\ S \end{tikzcd}$$

Category C/S objects of C lying over S.

If X is group object over S, $X_{S'}$ is group object over S'.

$$\begin{tikzcd} G \arrow[r, dash] \arrow[d, dash] & G_S \arrow[d, dash] \\ e \arrow[r, equal] & S \end{tikzcd} \qquad \begin{tikzcd} X \arrow[r, dash] & X_{S'} \arrow[d, dash] \\ S & S' \arrow[l] \end{tikzcd}$$

in affine case

$$\begin{tikzcd} A \arrow[d, dash] & \\ K \arrow[r] & K' \end{tikzcd} \qquad \mathrm{Hom}_K(A, K')$$

is group in natural way

Example

$$\begin{array}{c} E^1 = \mathrm{Spec}\, \mathbb{Z}[T] \\ \downarrow \\ e = \mathrm{Spec}\, \mathbb{Z} \end{array}$$

(affine space)

functor is $A \longmapsto A$ $(+)$.

$\mathbb{G}_a$ additive group scheme.

$\mathbb{G}_{a_S}$ additive group scheme over S

[& ring structure as well]

$E^1 = \mathcal{O}$. (as ring) "universal ring".

$\mathcal{O}_S$ — E / quasi-coherent sheaf over S.

S

$\mathbb{V}(E)$.

represents $S' \longmapsto \mathrm{Hom}_{\mathcal{O}_{S'}}(E_{S'}, \mathcal{O}_{S'})$

↑ is group.

∴ $V(E)$ is (commutative) group scheme over S,

is scheme of modules over $\mathcal{O}_S$.

$\mathcal{O}_S(S') = \Gamma(S', \mathcal{O}_S)$.

Lecture 6 Jan 29

affine group objects in category etc.

eg. G_a group scheme $\mathcal{O}$ (ring scheme) (over $\mathbb{Z}$).

$E^1_{\mathbb{Z}}$.

$$G_m \qquad S \longmapsto \Gamma(S, \mathcal{O}_S)^*$$
$$\cap$$
$$\Gamma(S, \mathcal{O}_S)$$
$$\|$$
$$G_a(S).$$

$\therefore$ G_m is subfunctor of G_a.

G_a represented by Spec $\mathbb{Z}[t]$

G_m represented by Spec $\mathbb{Z}[t, t^{-1}]$.

$$G_m \subset G_a$$

$\boxed{\text{open subscheme = complement of 0 section.}}$

$$GL(n) = G \quad \text{absolute group scheme (ie over } \mathbb{Z}).$$

$$\begin{array}{c} GL(n) \\ \big| \\ \mathbb{Z} \end{array}$$

$\Gamma(S, \mathcal{O}_S) = \text{ring}$

$$G(S) = GL(n, \Gamma(S, \mathcal{O}_S)) \subset M_n(\Gamma(S, \mathcal{O}_S))$$

contravariant functors $\longrightarrow$ groups.

$$S \longrightarrow M_n(\Gamma(S, \mathcal{O}_S)) \quad \text{is represented by}$$

$$\operatorname{Spec} \mathbb{Z}[(X_{ij})_{1 \le i,j \le n}] \quad (\text{ring scheme}).$$

$G(S)$ is represented by ~~open~~ subscheme

$$\operatorname{Spec} \mathbb{Z}[(X_{ij})_{1 \le i,j \le n}, \Delta^{-1}]$$

$$\Delta = \det(X_{ij}).$$

((some category)) (A any object)

$$\begin{array}{c} \mathcal{A} \in \hat{\mathcal{C}} \\ \cup \\ \mathcal{A}^* \end{array}$$

$$A(S)^* \subset A(S)$$

$$\begin{tikzcd} A^*(S) \arrow[d, "\text{def.}"] \\ A \end{tikzcd}$$

If A is representable, is A^* representable?

for x, $\exists$ y such that $xy = 1 = yx$.

$$\begin{array}{ccc} A \times A & \xrightarrow{f_1} & A \\ (x, y) & \longmapsto & xy \end{array}$$

$$\begin{array}{ccc} A \times A & \xrightarrow{f_2} & A \\ (x, y) & \longmapsto & yx \end{array}$$

f_1, f_2 natural transformation

$$\begin{tikzcd} A \\ e \arrow[u, "1"] \end{tikzcd}$$

e = final functor

$$\begin{tikzcd} A_1 \arrow[dr, hook, bend left] & & \\ & A \times A \arrow[r] & A \\ A_2 \arrow[ur] & & e \arrow[u] \end{tikzcd}$$

A_1 = fibre product in f_1

A_2 = fibre product in f_2.

Now set $A^{*} = A_1 \times_{A \times A} A_2$ (is A^{*} as above).

so if A representable, so is A^{*}. [if $\mathcal{C}$ is stable under fibre products]

eg if $\mathcal{A} = M_n$

M $\mathcal{A}_K$ sheaf of algebras over S (~~quasi-coherent~~) associative and unit. the which is locally free of finite type as a sheaf of modules.

$\downarrow$

K S

$\mathcal{A}_{S'} = f^{*}(\mathcal{A})$. $\mathcal{A}_{S'}$ is sheaf of algebras

$$S' \longmapsto \Gamma(S', \mathcal{A}_{S'})$$

(schemes over S) can be defined for sheaf of modules

$N \otimes_K K'$

This is a ring.

This functor is representable — $\mathbb{E}_2$

by $\mathbb{V}(\check{\mathcal{A}})$

$\mathrm{Hom}_{\mathcal{O}_S}(\mathcal{O}_{S'}, \mathcal{A}_{S'})$

$\| \mathbb{E}$ dual

$\mathrm{Hom}(\mathcal{A}^{\vee}_{S'}, \mathcal{O}_{S'})$

using locally free.

$= \mathbb{V}(A^\vee)(S')$

(def. p33)

$\underline{\mathcal{A}}$ sheaf of algebras.

$\cup$

$\mathcal{A}'$ subsheaf of algebras. Is ~~sub~~ ~~representable~~

(affine distinction) Spec M.

M (module K algebra)

$\mathcal{A}$ (commutative) quasi-coherent sheaf of algebras.

S' suitable? ...

S S

$X = \mathrm{Spec}(\mathcal{A})$ represents the functor

M... K algebra of M ...

S' ... over S

$S' \longmapsto \mathrm{Spec}(\mathcal{A})(S') = \mathrm{Hom}_{\mathcal{O}_S\text{-algebras}}(\mathcal{A}_{S'}, \mathcal{O}_{S'})$

S

contravariant functor $\mathcal{A}/S$ ... sets

Spec(A) is presumably a scheme over S?

(a) X is sheaf for Zariski topology clear.

(b) ~~if the natural~~ w.m.a. S affine.

$S = \operatorname{Spec} K$

$A \longrightarrow M$ alg over K. Then is represented by $\operatorname{Spec} M$.

(above is done in FGA I by [illegible] [illegible], above [illegible] [illegible] way). ?

[illegible] $\otimes_K K' \longrightarrow K'$ is same as

M q.c. sheaf of module.

$$V(M) = \operatorname{Spec}\left(\underline{\operatorname{Sym}}^*_{\mathcal{O}_S}(M)\right).$$

S

$\underline{\operatorname{Sym}}_{\mathcal{O}_S}(M)$

curve

• denotes ~~[illegible]~~ direct sum over [illegible] [illegible] [illegible] $\operatorname{Sym}(M)$ [illegible] [illegible]

$$\begin{array}{ccc} & & M \\ & & \downarrow \\ S' & \longmapsto & S \end{array}$$

$\mathcal{M}' \to \mathcal{O}_{S'}$ module hom

$\mathcal{E}_S'$

hom $\operatorname{Sym} M'$

$$A^* \subset \underline{A} = \mathbb{V}(A^\vee)$$

$$= \mathrm{Spec}\ \underline{\mathrm{Sym}}^*(A^\vee)$$

Δ = section of $\mathcal{O}_X$,

same as section of $\underline{\mathrm{Sym}}^*(A^\vee)$.

$$\Delta \in \Gamma(S, \underline{\mathrm{Sym}}^*(A^\vee)).$$

Note $\mathcal{O}_S(X) = \ldots\ E_S'$.

$$\stackrel{?}{=} \Gamma(X, \mathcal{O}_X).$$

$$X \longrightarrow \mathcal{O}_S = E_S'$$

$$X(S') \underset{(847)}{=} t \in \Gamma(S', A_{S'}) \xrightarrow{\text{seek}} \mathcal{O}_S(S') = \Gamma(S', \mathcal{O}_{S'})$$

$$N_{A_{S'}/\mathcal{O}_{S'}}$$

(tl. of translation) left or right? norm (defined in usual way)

$\mathcal{A}$ sh of $\mathcal{O}_S$-algebras over S. [[illegible]]

$S = \bigcup U_i$ $\qquad$ $\mathcal{A}|U_i = A_i$.

$\;\;\;\;\; \Vert$ Spec S_i

$$\begin{array}{c} \operatorname{Spec} A_i \\ \downarrow \\ U_i \end{array}$$

$\Delta_i \in A_i$. The Δ_i's patched to give section of $\mathcal{A}$, Δ

<u>local case</u>. $M \ni x$

K (local ring) [reduce to field by Nakayama].

?

<u>section of</u>

(ς: mm) $A^* \subset X \xrightarrow{\Delta} \mathcal{O}_S = E_S^1$

$$X_\Delta = \operatorname{Spec}\left((\operatorname{Sym}^* A^\vee)_\Delta\right)$$

$\|$

A^*.

efficient way of getting

A^* as Spec of something.

<u>Try affine case</u>. maybe more concrete.

M locally free sheaf of S modules of finite type.

$|$

S

$\underline{Aut}_{\mathcal{O}_S}(M)$

functor on schemes over S.

$$\begin{array}{ccc} M_{S'} & \to & M \\ | & & | \\ S' & \to & S \end{array}$$

~~$S' \to S$~~

$$\underline{Aut}_{\mathcal{O}_S}(M)(S') = \underline{Aut}_{\mathcal{O}_{S'}}(M_{S'}).$$

Is representable by affine scheme over S.

representable - (vector bundle).

by $\mathbb{V}(M \otimes M^\vee)$ — S'

$\operatorname{End}_{\mathcal{O}_{S'}}(M_{S'})$.

ring

$$\operatorname{End}(M) \cong M^\vee \otimes M \qquad M^\vee \otimes M|_{S'} = M^\vee_{S'} \otimes M_{S'}$$

$\therefore\ \underline{Aut}_{\mathcal{O}_S}(M)$ is represented as spectrum as above

$$\underline{\mathrm{Sym}}^*(M \otimes M^\vee)$$

Δ

Can replace by Δ instead $\Delta \in \underline{\mathrm{Sym}}^n(M \otimes M^\vee)$.

instead of general S in $\underline{\mathrm{Sym}}^{m^2}(M \otimes M^\vee)$.

$\mathcal{C}$ G group scheme, acts on something.

G — group object in $\mathcal{C}$ (category).

$X \in \hat{\mathcal{C}}$.

G acts on X

means $\forall S$, $G(S)$ acts on $(X)(S)$ in functorial way.

If X has pointed structure $(e \xrightarrow{\varepsilon} X)$.

Then can talk about subgroup leaving ε fixed.

$$H(S) \subset G(S)$$

$$\{ h \in G(S) \mid h \cdot \varepsilon(S) = \varepsilon(S) \}$$

If G representable, H may be also. (X representable too). eg if $\mathcal{C}$ has final objects and fibre products.

$g\cdot\mathcal{E} = \mathcal{E}$.

$$\begin{tikzcd} H \arrow[r, hook] & G \arrow[r, shift left, "g \mapsto g\cdot\mathcal{E}"] \arrow[r, shift right, "g \mapsto \mathcal{E}\text{ constant}"'] & X \end{tikzcd}$$

where arrows coincide.

(define elementwise).

finite inverse limit

$$\begin{tikzcd} G \arrow[r] \arrow[d] & X\times X \\ \cdot \arrow[r] & X \arrow[u, "\mathrm{diag}"'] \end{tikzcd}$$

(X representable)
(or diagram in $\mathcal{C}$)

is fibre product

$\mathcal{O}_S$ group functor.
group scheme.

G

$|$

S scheme.

$\mathcal{M}$ sheaf of modules.

G acts on $\mathcal{M}$.

Naturally in S' want $S' \mapsto \Gamma(S', \mathcal{M}_{S'})$
compatible over $\Gamma(\mathcal{O}_{S'})$?

underlying functor and $\mathcal{M}$.

S'

acts on $\mathcal{M}$ respecting module structure.

eg GL_n acts on K^n. ~~SL_2 acts~~

$\underline{\mathrm{Aut}}_{\mathcal{O}_S}(M)(S')$ acts on M.

I say G operates on M means $\exists$ hom. G

If G acts on M, then it acts on $M^\vee$, tensor algebra, exterior algebra etc. (transport of structure) [only in case of iso, which commute with base change].

If M locally free, ... (?)

if have section τ can talk of subgroup functor leaving the section fixed. G^τ.

Section of $M \otimes M$ is quadratic form on $M^\vee$.

can talk about orthogonal group, alternating group etc. Subgroups of GL_n.

$\exists$ representable $\rightarrow$ if affine scheme. (Thm of Chevalley).

affine group scheme of finite type over k

Thm. of Chevalley

$$\begin{array}{ccc} B & & Y \\ & & | \\ A & & Y \end{array} \qquad \begin{array}{c} G \\ | \\ K \end{array}$$

~~aff~~ group scheme of finite type. (K field) any

Then G is affine $\Leftrightarrow$ $\exists$ a monomorphism [turns out to be closed immersion]

$G \hookrightarrow Gl(m)_K$ some m.

(i.e. no talk about leaving the manifold).

Feb 3

$$\begin{array}{c} X \\ \downarrow \\ S \end{array}$$

X affine over S

Equivalent are

① $X = \operatorname{Spec} \mathcal{M}$ — $\mathcal{M}$ quasi-coherent sheaf of algebras on S.

② $\forall\, U \subset S$ affine — $X|U$ is affine (i.e. is $\operatorname{Spec} B$, B an A-algebra, $U = \operatorname{Spec} A$)

(enough to check for U_i's covering X)

$$\begin{array}{ccc} f^{-1}(U) & \longrightarrow & X \\ \downarrow & & \downarrow \\ U & \longrightarrow & S \end{array}$$

① $\Rightarrow$ ② clear.

② $\Rightarrow$ ①. patch together.

If S affine, then X affine over S $\Longleftrightarrow$ X is affine

All above group schemes are affine. ?

G_S constant scheme with value S.

$$G \text{ set}) \quad \coprod_{g \in G} S_g \quad \overset{\text{def}}{=} \quad \begin{array}{c} G_S \\ \downarrow \\ S \end{array} \text{ scheme.}$$

(disjoint union)

discrete.

$$\coprod_g S_g = S' \nearrow S$$

Functor wise

$$S' \longrightarrow \mathrm{Hom}_{\text{Schemes}}(S', G_S)$$

just ~~set~~ scheme

$$= \text{Continuous map}\,(S', G).$$

(locally constant maps).

$$G_S = (G_{\mathbb{Z}})_S$$

$$G \longmapsto G_{\mathbb{Z}}$$

$$G_S$$

left exact functor.

If G is a group, get group scheme etc.
ring ring scheme to

(clear if look at the functor)

G infinite $\Longrightarrow$ G_S not affine
S affine (since not compact).
<u>not of finite type either.</u>

Elliptic curve — not affine

$$\begin{array}{c} X \\ \Big| \\ k \end{array} \quad (\text{alg. closed}).$$

if not alg closed modify definition slightly.

$Pic(X)$ = classes of locally free sheaves rank 1 $\boxed{\text{is group}}$ with $\otimes$ $\boxed{\text{any ringed space even}}$

~~$Pic(X)$~~

$Pic_{X/k} : (\text{Schemes}/k) \longrightarrow$ abelian groups.

("a family of schemes on X parametrized by S")

$$\begin{tikzcd} X \arrow[d] & X_S \arrow[l] \arrow[d, "f"'] \\ \operatorname{Spec} k \arrow[r, equal] & S \end{tikzcd}$$

$$\frac{\operatorname{Pic}(X_S)}{f^*(\operatorname{Pic}(S))} \overset{\text{def.}}{=} \operatorname{Pic}(X/S)$$

in the case of $S = k$, set $\operatorname{Pic}(X) = \text{set of } k\text{-valued points of } \underline{\operatorname{Pic}}_{X/k}$.

$$\begin{tikzcd} \operatorname{Pic}(X_S) \arrow[r] & \operatorname{Pic}(X_{S'}) \\ \operatorname{Pic}(S) \arrow[u] \arrow[r] & \operatorname{Pic}(S) \arrow[u] \end{tikzcd}$$

$S \longrightarrow \operatorname{Pic}(X/S)$ is the functor.

$\underline{\operatorname{Pic}}_{X/k}$. representing object.

Is this representable? — ~~yes if you assume~~

Theorem (Murre) If X/k is proper, then $\underline{\operatorname{Pic}}_{X/k}$ is representable. Connected component $\underline{\operatorname{Pic}}^0_{X/k}$ is of finite type.

constant group scheme.

$$\underline{\mathrm{Pic}}_{X/k} \Big/ \underline{\mathrm{Pic}}^0_{X/k} \cong \underline{NS}_{X/k} \text{ (Neron Severi grp)}$$

Presumably same [illegible] to any S.

image [illegible] quotient grp $\dfrac{\underline{\mathrm{Pic}}_{X/k}(S)}{\underline{\mathrm{Pic}}^0_{X/k}(S)}$.

[illegible] if S [illegible] grp [illegible] finite type.

N is f.g. $\mathbb{Z}$-module of finite type (Neron) ~~[illegible]~~ eg [illegible] argument (F-Severi)

(could have torsion).

~~protector~~

If X non-singular

$\underline{\mathrm{Pic}}^0(X/k)$ is proper [illegible] group scheme (hence can't be affine)

X alg curve $\to$ $\underline{\mathrm{Pic}}^0_{X/k}$ is abelian variety = Jacobian.

Elliptic curve $\cong$ its own $\underline{\mathrm{Pic}}^0_{X/k}$.

k alg. closed $\Longrightarrow$ only 1 dim'l non singular connected group schemes finite type are G_a, G_m, elliptic curves.

(Chevalley - seminar on group schemes. 1956)

No homomorphisms $G_a \to G_m$.

(esp in analysis in infinite series)

k alg. closed (perfect is O.K. too)
G connected smooth (~~commutative~~ $\boxed{\text{not necessary}}$) group scheme.

$$0 \to \underset{\text{affine}}{L} \to G \to \underset{\text{abelian variety}}{A} \to 0$$

k perfect.

$$\underset{\text{affine commutative}}{L} \cong L_n \times \underbrace{G_m^r}_{\text{torus}}$$

(if not alg closed torus could be twisted?)

L_n: unipotent ($\exists$ composition series with quotients G_a)

(see Serre $\boxed{\text{Borel}}$)

If L commutative, then Z in G
(exercise!)

$$G \times G \longrightarrow 1 \times A$$
$$(x,y) \longmapsto xyx^{-1}y^{-1}$$

(but it remains the argument!)

If L not commutative
Z centre (need not be connected)

$$0 \to L \longrightarrow G \longrightarrow A \to 0$$

centre $\cup$ $\quad$ $\cup$ $\quad$ H $\quad$ Z

$$0 \longrightarrow L \longrightarrow G \longrightarrow A \longrightarrow 0$$

$\cup$ $\quad$ centre $\cup$ $\quad$ Z $\quad$ epi.

$Z \cap L$

reference SGA 3
exposé
exposé
groupes
groupes

Will only speak on affine group schemes.

μ_n $\quad$ n^{th} roots of unity

$$\mu_n \longrightarrow G_m \xrightarrow{\lambda \to \lambda^n} G_m$$

- presumably we want to think that morphisms of schemes which are ~~[illegible]~~ group homomorphisms on functors of points

$$\begin{tikzcd} K \arrow[r, hook] & G \arrow[r] \arrow[dr] & H \arrow[d] \\ & & S \end{tikzcd}$$

$$K(S') = \ker \left(G(S') \longrightarrow H(S') \right)$$

subgroup.

<u>subgroup functor.</u>

If ~~fibre~~ fibre products representable (as schemes)

Then kernel is representable.

$$\mu_n(S) = \ker \quad G_m(S) \longrightarrow G_m(S)$$
$$\lambda \longrightarrow \lambda^n.$$

$$\| $$

$$\{ \lambda \in \Gamma(S, \mathcal{O}_S) \mid \lambda^n = 1 \}$$

Kernel in affine case

$$G \subset H$$

$$A \longleftarrow B \supset I_B \quad \text{(kernel of } \varepsilon\text{)}.$$

map of rings.

ε unit section

K i

Kernel is $\operatorname{Spec}\left(\frac{A}{I_B A}\right)$.

eg

$$0 \to \mu_n \longrightarrow G_m \longrightarrow G_m \longrightarrow 0$$

$$\mathbb{Z}[T, T^{-1}] \longleftarrow \mathbb{Z}[T, T^{-1}] \quad (T \to 1 \text{ under } \varepsilon.)$$

$$T^n \longleftarrow T$$

$$T^n - 1 \leftarrow T - 1$$

Augmentation ideal ~~Z~~ gen by $T-1$.

get $\operatorname{Spec} \frac{\mathbb{Z}[T, T^{-1}]}{(T^n - 1)} = \operatorname{Spec} \frac{\mathbb{Z}[T]}{(T^n - 1)}$

(rings iso.)

can get also by direct definition.

Def finite locally free group schemes.

$= \operatorname{spec}(A)$ where A is a sheaf of algebras which as sheaf of modules is locally free of finite type.

$n = n'n''$ $\qquad \mu_n \cong \mu_{n'} \times \mu_{n''}$

$(n', n'') = 1$ as group scheme.

${}_nG = 0 \qquad {}_nG(S) = 0$

$G(S) \cong \underbrace{{}_{n'}G(S)}_{\text{stuff killed by } n'} \times \underbrace{\boxed{{}_{n''}G(S)}}_{\text{stuff killed by } n''}$

canonical.

${}_{nn'}G$, G any Group scheme. nn' $(n, n') = 1$

$G = {}_nG \times {}_{n'}G$ $\qquad {}_nG, {}_{n'}G$ representable

(category stable under fibre products)

${}_{n'}\mu_n = ?$

$$\left(\mu_{p^r}\right)_{k} = \begin{cases} p \neq \operatorname{char} k. & \left(\mu_{p^r}\right)_k = \text{constant group } \left(\mathbb{Z}/p^r\mathbb{Z}\right)_k \\ p = \operatorname{char} k & \left(\mu_{p^r}\right)_k \text{ reduced to one point as scheme.} \\ & \operatorname{Spec}\left(k[t]/t^{p^r}\right) \text{ (nilpotent elements here)} \end{cases}$$

k alg closed.

base change.

ie kernels need not be reduced schemes

"geometric fibres"

$\operatorname{Spec}(k) = pt.$

OK for alg closed field.

If not alg closed - richer structures.

$\mu_2 = O(1)$ — mult by λ, $\lambda^2 = 1$

orthogonal group.

in char 2, (purely infinitesimal group).

Feb 5

$$x \longrightarrow x^n$$

$$0 \to \mu_n \longrightarrow G_m \longrightarrow G_m \to 0$$

exact sequence.

"Kummer exact sequence".

epimorphism here not defined yet. More than just epi in category of schemes.

Def. of exactness here clear.

μ_n can be decomposed

$$\mu_n = \prod \mu_{p_i^{r_i}}$$

$n = \prod p_i^{r_i}$ prime powers

$$\mu_{p^r} = \begin{cases} \cong (\mathbb{Z}/p^r\mathbb{Z})_k & \text{if prime to char.} \\ \operatorname{Spec} k[t]/t^{p^r} & p = \text{char.} \end{cases}$$

X top space

$X \neq \emptyset$ irreducible if, whenever $X = X_1 \cup X_2$, X_i closed

$\Rightarrow X_1 = X$ or $X_2 = X$.

otherwise X reducible.

$\therefore$ Y irreducible $\Longrightarrow$ $\overline{Y}$ irreducible.
(induced topology)

$\therefore$ ~~every~~ ~~X is a cov~~ X is union of max. irreducible (∃ by Zorn).
closed subsets = irreducible components of X.

$$X = \bigcup_{i \in I} X_i \qquad I \text{ could be infinite}$$

If X noetherian (for closed sets).
$\Rightarrow$ I finite

eg

eg A noetherian $\Rightarrow$ Spec A noetherian

converse false. eg $k \oplus V$ V infinite dim'l v.s.
$v^2 = 0$

bij Spec A $\leftrightarrow$ ~~2~~ ideals J such that A/J has no nilpotent elements.

~~def~~ closed irreducible subsets of Spec A have unique generic points.

bijection pts $\leftrightarrow$ irreducible closed subsets.

also for schemes (more generally — for any space covered by fin sets having this property) [illegible] — every pt has a quasi-compact neighborhood (eg any affine neighborhood).

Irreducible components of μ_{p^r}.

$r+1$ of them.

$$\begin{tikzcd} X \arrow[d, "\text{flat}"] \\ Y \end{tikzcd}$$

every irreducible compnt of X dominates irreducible compnt of Y

$$\begin{tikzcd} \mu_{p^r} = \mathrm{Spec}\,\dfrac{\mathbb{Z}[t]}{(t^{p^r}-1)} \arrow[d] & \mathrm{Spec}\,\dfrac{\mathbb{Q}[t]}{(t^{p^r}-1)} \arrow[l] \arrow[d] \\ \mathrm{Spec}\,\mathbb{Z} & \mathrm{Spec}\,\mathbb{Q} \arrow[l] \end{tikzcd}$$

$$(t^{p^r}-1) = \prod_{1\le i\le p^r} (t-\zeta^i) = \Phi_{p^r}\,\Phi_{p^{r-1}} \cdots \Phi_1, \quad \Phi_1 = t-1.$$

ζ = primitive p^r-th root of unity (in alg. closure)

product of those ζ^i of a given order is cyclotomic polynomial.
(each irreducible over $\mathbb{Q}$ — Kronecker remarkable!)

These correspond to irreducible components over $\mathbb{Z}$.

$$\underbrace{\mu_{p^r}^{*}}_{\text{scheme of primitive } p^r\text{-th roots of unity}} \subset \mu_{p^r}.$$

$$\mu_{p^r}^{*}(S) = \left\{ \lambda \in \Gamma(S, \mathcal{O}_S) \;\middle|\; \begin{array}{l} (a)\ \lambda^{p^r}=1 \\ (b)\ \forall s \in S \\ \quad \lambda(s)^{p^{r-1}} \neq 1 \end{array} \right\}$$

open subset — but not group scheme (over 1).

$$\mu_{p^r} = \bigcup_{0\le i\le r} \mu_{p^i}^{*}$$

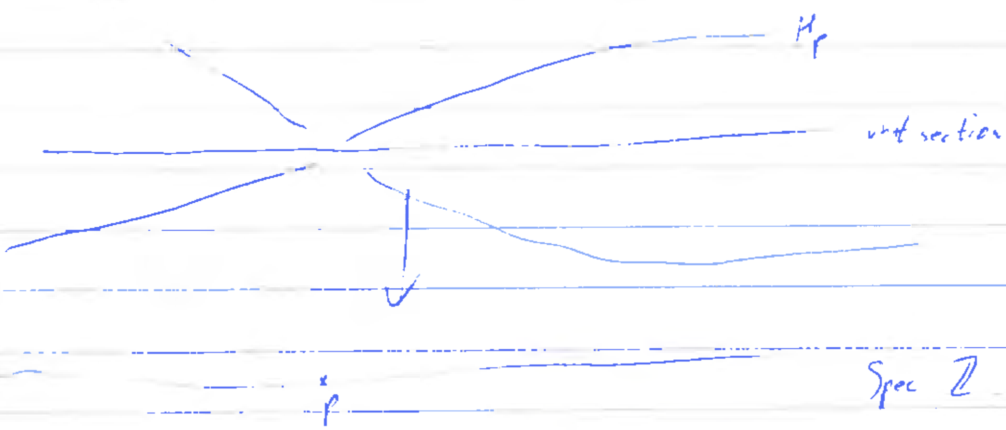
disjoint union of locally closed subsets.

(closure of each gives the irreducible components) $\overline{\mu_{p^i}^{*}}$.

$$\mu_{p^i} \xrightarrow[\text{closed.}]{} \mu_{p^r} \qquad i<r.$$

ex $\quad \mathcal{M}_p = \mathcal{M}_p^* + \mathcal{M}_s^*$

unit section.



$\overline{\mathcal{M}_{p^i}^*}$ all meet at the unique point of $\mathcal{M}_{p^r}$ which lies over p [each is mapped ~~surjectively~~ to Spec $\mathbb{Z}$]

- image dense (since flat).
- proper morphism.

Fact $\quad X \xrightarrow[f]{\text{finite}} S \quad$ i.e. $X = \operatorname{Spec} \mathcal{A}$ $\quad$ ($\mathcal{A}$ sheaf of modules of finite type.)

Then f is proper. (Cohen - Seidenberg).

another Example of proper morphism

$$\begin{array}{c} \mathbb{P}^r_S \\ \downarrow \text{proper} \\ S \end{array}$$

closed subschemes of $\mathbb{P}^r_S$ are proper.

over any ground field

char p

$\operatorname{Spec} \dfrac{k[s]}{s^{p^r}}$

$s = t-1$

nonpoint

char $k \neq p$.

$$\begin{array}{c} \mu_{p^r} \\ | \\ k \end{array}$$

(étale

finite union of spectra of ~~spectra of finite~~ separable extensions.

$k \xrightarrow[\pi \text{ galois group}]{} \bar{k}$

$X \longrightarrow X(\bar{k})$

big étale schemes$/k$ $\xleftrightarrow{\text{equivalence of categories}}$ finite sets on which π operates continuously. [π can be infinite of course.]

"Galois theory"

π acts [illegible] on the group of $\bar{K}$ [illegible] points, and [illegible] the points of μ_{p^r}

μ_{p^r} — $\mathbb{Z}/p^r\mathbb{Z}$ ring

autos. are invertible elts. = integers prime to p.

π acts [illegible]

$\operatorname{Spec} K \longleftarrow \operatorname{Spec} \bar{K}$

$$\gamma \in \pi \xrightarrow{\phi_r} \left(\frac{\mathbb{Z}}{p^r\mathbb{Z}}\right)^*$$

$$\xi^{\phi_r(\gamma)} = \gamma(\xi)$$

$$\mu_{p^r} \hookrightarrow \mu_{p^s} \qquad r < s$$

ϕ_s

Inverse system

$$\begin{tikzcd} \pi \arrow[dr, "\phi_r"] \arrow[ddr, "\phi_s"'] & \\ & \left(\frac{\mathbb{Z}}{p^r\mathbb{Z}}\right)^* \\ & \left(\frac{\mathbb{Z}}{p^s\mathbb{Z}}\right)^* \arrow[u] \end{tikzcd}$$

$$\phi_p : \pi \longrightarrow (\mathbb{Z}_p)^*$$

[illegible] for all μ_{p^r}.

Answer (in terms of image of π in $(\mathbb{Z}_p)^\wedge$)

$= H$ (closed subgroup).

Look at orbits of H on $\mu_{p^r}(\bar{k}) \cong \mathbb{Z}/p^r\mathbb{Z}$

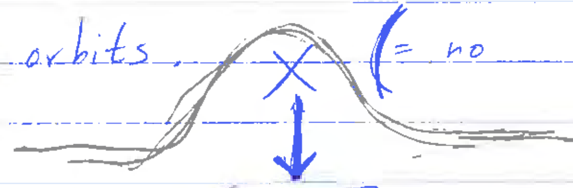
Take $\mathbb{F}_\ell$ — $G = \hat{\mathbb{Z}}$ (gen by absolute frobenius).

$\hat{\mathbb{Z}} \longrightarrow \mathbb{Z}_p^\times \ni \ell$ (image).

~~exponentiate to ℓ^{th} power in~~

multiply by ℓ in $\mathbb{Z}/p^r\mathbb{Z}$

and count orbits. (= no of components)



Example $\quad \operatorname{Spec} \mathbb{F}_p \longrightarrow \operatorname{Spec} \mathbb{Z}$.

schemes of char p.

$\lambda \longrightarrow \lambda^p$ is ring homomorphism in $\mathcal{O}_X(U)$ (gives endo of $\mathcal{O}_X$).

$\operatorname{Spec} \mathbb{F}_p$ final object.

must check gives local homomorphism on the stalks.

$X \xrightarrow{f_X} X$ over $\mathbb{F}_p$

absolute frobenius is identity on points

is functorial — ie

ie
$$\begin{tikzcd} X \arrow[r, "\mathcal{F}_X"] \arrow[d] & X \arrow[d] \\ Y \arrow[r, "\mathcal{F}_Y"] & Y \end{tikzcd}$$

$\mathcal{F}$ = endomorphism of identity functor on category of schemes over $\mathbb{F}_p$.

also p^r ~~[illegible]~~ th power.

$$\begin{tikzcd} G_{\mathbb{F}_p} \arrow[r, "\mathcal{F}_G"] \arrow[d] & G \arrow[dl] \\ \mathbb{F}_p. & \end{tikzcd}$$

G group scheme.

Then $\mathcal{F}_G$ is group homomorphism.

eg G_a

$$S \longrightarrow \Gamma(S, \mathcal{O}_S)$$

$$\lambda \longrightarrow \lambda^p.$$

$$\alpha_p \longrightarrow G_a \longrightarrow G_a$$

$$\lambda \longrightarrow \lambda^p$$

Kernel of $\lambda \to \lambda^p$.

$$\lambda \longrightarrow \lambda^{p^n}$$

Kernel is α_{p^n}.

$$\alpha_{p^n}(S) = \{ s \in \Gamma(S, \mathcal{O}_S) \mid \lambda^{p^n} = 0 \}.$$

$$\alpha_{p^n} \cong \mathrm{Spec}\, \frac{\mathbb{F}_p[T]}{(T^{p^n})}$$

$\boxed{\text{purely infinitesimal}}$

$\cong$ as scheme to μ_{p^n}.

but group structures different.

Feb 10 S is of char. 0 if S lies over Spec $\mathbb{Q}$.

$$\begin{tikzcd} & S \arrow[d] \arrow[dl, dashed] \\ \operatorname{Spec}\mathbb{Z} & \operatorname{Spec}\mathbb{Q} \arrow[l, hook'] \end{tikzcd}$$

$\mathbb{Q} \hookrightarrow \Gamma(S, \mathcal{O}_S)$. algebra structure over $\mathbb{Q}$.

$\mathcal{O}_S$ = sheaf of $\mathbb{Q}$-algebras.

Need not have a characteristic eg Spec $\mathbb{Z}$.

Bourbaki char R = ker $\mathbb{Z} \to R$.
(p) $n \longrightarrow n \cdot 1_R$.
p = char R.
not good since all characteristics built in.

$\operatorname{Spec} \dfrac{\mathbb{Z}}{p^n\mathbb{Z}}$ no characteristic if $n \geq 1$.

$\downarrow$

Spec $\mathbb{Z}$

Remark (Frobenius)

$\mathcal{C}$ category

$f : id \longrightarrow id$ end. of identity functor.

$X \times X \xrightarrow{\pi} X$

$f_X : X \longrightarrow X$ f_X compatible with multiplication.

etc.

f gives endo. of group object, ring object, etc.

Translations into affine case

$$\begin{array}{ccc} G \text{ affine} & & A \\ \downarrow & & \uparrow \\ S = \operatorname{Spec} k & & k \end{array}$$

A: algebra over k.

corresponds to

$G \times_S G \xrightarrow{\pi} G$ composition

$A \otimes_k A \xleftarrow{\Delta} A$ k-algebra homomorphism.

"diagonal map". comultiplication

$\text{ox } \cdot \longrightarrow k$

$A \otimes B \longleftarrow C$

"bialgebra"

commutative

$[(x,y) \longrightarrow (y,x)$ in functors$)]$

$$\begin{tikzcd}
G \times_S G \arrow[rr] \arrow[dr, "\pi"'] & & G \times_S G \arrow[dl, "\pi"] \\
& G &
\end{tikzcd}$$

commutes.

$y \otimes x \longleftarrow x \otimes y$

$$\begin{tikzcd}
A \otimes_k A & & A \otimes_k A \arrow[ll] \\
& A \arrow[ul, "\Delta"] \arrow[ur, "\Delta"'] &
\end{tikzcd}$$

commutes.

$$\delta(x) = \sum y_i \otimes z_i$$

$$= \sum z_i \otimes y_i$$

associativity.

$(x, (y,z)) \longrightarrow (x, yz)$

$$\begin{tikzcd}
G \times G \times G \arrow[r, "1 \times \pi"] \arrow[d, "\pi \times 1"'] & G \times G \arrow[d, "\pi"] \\
G \times G \arrow[r, "\pi"'] & G
\end{tikzcd}$$

$(xy, z) \longrightarrow xyz.$

commutes

$(\Delta x) \otimes y$

$$\begin{tikzcd}
A \otimes A \otimes A & & A \otimes A \arrow[ll, "1_A \otimes \delta"'] \\
A \otimes_k A \arrow[u, "\delta \otimes 1_A"] & & A \arrow[ll, "\Delta"] \arrow[u, "\Delta"']
\end{tikzcd}$$

$x \otimes y$

commutes

S = final object in schemes over S.

$$\begin{tikzcd} G \\ S \arrow[u, "\varepsilon"] \end{tikzcd}$$

ε = unit

$$\begin{tikzcd} G \arrow[r, "\varepsilon \times 1_G"] \arrow[dr, "1"'] & G \times G \arrow[d, "\pi"] \\ & G \end{tikzcd}$$

[ε is left unit]

commutes.

$$\begin{tikzcd} A \arrow[d, "\varepsilon"] \\ k \end{tikzcd}$$

algebra homomorphism

"augmentation"

translates to:

$$\begin{tikzcd} k \otimes_k A & A \otimes_k A \arrow[l, "\varepsilon \otimes \mathrm{id}"'] \\ & A \arrow[ul] \arrow[u, "\Delta"'] \end{tikzcd}$$

commutes.

Similarly for right unit.

"unit" = two-sided unit.

> Can do above if A only k-module — haven't used the algebra structure

$$G \xrightarrow{\alpha} G$$

inverse

$$A \xleftarrow{a} A$$

a = algebra homomorphism.

$$\begin{tikzcd} G \arrow[r, "(1_G, a)"] \arrow[d, "\mathrm{can.}"'] & G \times G \arrow[d, "\pi"] \\ S \arrow[r, "\varepsilon"] & G \end{tikzcd}$$

a automorphism of G (in reasonable case).

$$\begin{tikzcd} A & A \otimes_k A \arrow[l, "(\mathrm{id}_A, a)"'] \\ k \arrow[u, "\mathrm{can.}"] & A \arrow[l, "\varepsilon"'] \arrow[u, "\delta"'] \end{tikzcd}$$

$\exists$ inverse.

$a^2 = 1$.

Bialgebras – first met in alg. topology.

G topological group.

$H^*(G, k)$ ~~has~~ is graded ring over k.

(anticommutative $xy = (-1)^{ij} yx$

i = degree x

j - degree y.

$$G \times G \xrightarrow[\text{mult.}]{\delta} G$$

$$H^*(G \times G) \xleftarrow{\delta^*} H^*(G)$$

canonical. As graded algebras. $\cong \|$ G compact say. sometimes true ~~also~~ other cases

$$H^*(G) \otimes_K H^*(G)$$

δ associative $\Rightarrow$ δ^* associative etc.

tensor product.

Similarly unit in G gives "unit" $H^*(G)$

$$G \underset{e}{\downarrow} \qquad H^*(G) \downarrow K$$

unit is section of unique map $G \to e$.

etc.

equivalent

$$\begin{array}{l} G \quad \text{affine over } S \\ \downarrow \\ S \quad \text{any scheme} \end{array}$$

$\mathcal{A}$ q.c. sheaf of algebras over S.

Algebra structure is notation $A \otimes_{\mathcal{O}_S} A \to A$.

Above translate to corresponding maps $\mathcal{A} \otimes_{\mathcal{O}_S} \mathcal{A} \xleftarrow{\Delta} \mathcal{A}$ for composition law, etc.

eg augmentation

$$\mathcal{A} \xrightarrow{\varepsilon} \mathcal{O}_S$$

"sheaf of ~~bialgebras~~", Bialgebra, bialgebra

Sometimes write "algebra" ~~instead~~ of sheaf of algebras.

Assume $G \overset{= \operatorname{Spec}(A)}{}$ has monoid structure

$\underline{M}$ module ~~set~~ functor.

(~~defined on category of K-algebras~~).

$S = \operatorname{Spec} k$

(p47)? module functor is represented by $V(\underline{M}^\vee)$? maybe only a Krull free case?

$\underline{M}$ = functor corresponding to M.

G acts on M (linearly)

$$\underline{M}(K') \equiv M \otimes_K K'.$$

$$G \times M \longrightarrow M.$$

(over $S = \operatorname{spec} K$).

K'.

$g \in G(K')$.

$$\underline{O}(K') = K'.$$

$\underline{M}(K')$ is module over K'.

"module functor".

G monoid functor. $\forall K'$. $G(K')$ acts on $M(K')$ by identity. functorial in K'.

$G = \operatorname{Spec}(A)$ represented by algebra over K.

$\underline{M}$ by " " M

$$G \times \underline{M} \longrightarrow \underline{M}$$

linear action

$$A \otimes_K M \xleftarrow{\omega} M$$

homo. of K-modules.

$g: A \to K'$. Take $K' = A$, $g = \mathrm{id}$.

$g_{K'}: M \otimes K' \longrightarrow M \otimes_K A$ homo of A-modules here

$M \to A$ same as K-modules $M \to M \otimes_K A$.

$\mathcal{O}(G) = A$

$$\begin{tikzcd} & A \arrow[dr] & \\ G \arrow[d, dash] & & K' \\ k \arrow[r, dash] & S' = \operatorname{Spec} K' \arrow[ul] & \end{tikzcd}$$

$M \otimes_K A \to M \otimes_K A$

$(M \otimes_K A) \otimes_A K' \to (M \otimes_K A) \otimes_A K'$

$M \otimes_K K' \longrightarrow M \otimes_K K'.$

$K'.$

Assume A bialgebra

Associative law. Translation

$$\begin{tikzcd} G \times (G \times \underline{M}) \arrow[r, "id \times w"] \arrow[d, "\pi \times 1"] & G \times \underline{M} \arrow[d, "w"] \\ G \times \underline{M} \arrow[r, "w"] & \underline{M} \end{tikzcd}$$

$$\begin{tikzcd} A \otimes_K (A \otimes_K M) & A \otimes_K M \arrow[l, "id \otimes w"'] \\ A \otimes_K M \arrow[u, "\Delta \otimes 1_M"] & M \arrow[l, "w"] \arrow[u, "w"'] \end{tikzcd}$$

"commutativity" = comodule over coalgebra over A.

commutativity of 2 diagrams equivalent.

check its $K' = A \otimes A$ "universal case"

since $\downarrow$

1 pts of S' in G

is one pt in $G \times G$.

[illegible]

Feb. 11

$$\begin{array}{c} G \\ | \\ k \end{array}$$

$$A \otimes_k A \xrightarrow{\pi} A$$

algebra structure

various actions possible

$$A \xrightarrow{\Delta} A \otimes_k A \qquad \text{~~[illegible]~~ coalgebra structure.}$$

Various actions possible. (Some make sense without any π).

M module over k. 2 functors on algebras over k.

$$\underline{M}(K') = M \otimes_k K' \qquad \text{covariant in } M.$$

$\underline{M}$ not representable in general by Yoneda

K if M locally free

The big $V(M^\vee)$.

[illegible] $\Leftrightarrow$ M locally free

$$\overline{V}(M)(K') = \mathrm{Hom}_K(M, K') \qquad \text{cont. in } M$$

$$\cong \mathrm{Hom}_{K'}(M \otimes_k K', K')$$

$\underline{M}$ is module functor over $\mathcal{O} : K' \to K'$ [illegible]

$\mathcal{O}$: K-algebras $\to$ rings. represented by $\mathrm{Spec}\ K[t]$

$M \longmapsto \underline{M}$ fully faithful.

k-modules $\longrightarrow$ $\mathcal{O}$-modules (module functor over $\mathcal{O}$).

$M \longrightarrow \underline{M}$

fully faithful $:$ $\mathrm{Hom}_K(M,N) \xrightarrow{\sim} \mathrm{Hom}_{\mathcal{O}}(\underline{M},\underline{N})$.

$$M \times N \longrightarrow P \text{ bilinear} \overset{\text{bij}}{\longleftrightarrow} M \otimes N \to P$$

$\underline{M} \to \underline{N}$ evaluate on K to go back.

$\updownarrow$ by ~~...~~ given $\updownarrow$ bij

$$\hookrightarrow \underline{M} \times \underline{N} \longrightarrow \underline{P} \text{ bilinear also. } \underline{M} \otimes_{\mathcal{O}} \underline{N} \to \underline{P}.$$

Alg. structure on M is $M \otimes M \longrightarrow M$

Same as alg. structure on $\underline{M}$. (over $\mathcal{O}$).

$M \longrightarrow V(M)$ contravariant in M. module functor over $\mathcal{O}$ also.

$\mathrm{Hom}_K(M,N) \xrightarrow{\sim}$ ~~$\mathrm{Hom}(\underline{N},\underline{M})$~~

$\mathrm{Hom}_{\mathcal{O}}(V(N), V(M))$.

$$\mathcal{M}od(K)^{o} \longrightarrow \mathcal{M}od(\mathcal{O})$$

fully faithful.

from $M^\vee = \mathrm{Hom}_k(M, k)$ cannot recover M.

$\mathbb{V}(M)$ scoped up dual, more information allows you to recover M.

(EGA I ...)

$$\sigma : \mathbb{V}(M) \times \mathbb{V}(N) \xrightarrow{f} \mathbb{V}(P)$$ morphism of (affine) schemes.

$$\mathrm{Sym}_k^{\cdot}(M) \otimes \mathrm{Sym}_k^{\cdot}(N) \longleftarrow \mathrm{Sym}_k(P)$$ corr. map on rings

or

$$\mathrm{Sym}_k(M) \otimes \mathrm{Sym}_k(N) \xleftarrow{k\text{-linear}} P$$

$$\coprod_{i,j} \mathrm{Sym}^i(M) \otimes \mathrm{Sym}^j(N) \xleftarrow{k \text{ linear}} P$$

what does it mean for f to be bilinear?

$\mathbb{V}(M) \otimes_k \mathbb{V}(N) \longrightarrow \mathbb{V}(P)$

f bilinear $\iff$ P mapped entirely into $i=1, j=1$ piece.

$$M \otimes N \xleftarrow{k\text{-linear}} P$$

Take M, N, P equal $= M$.

algebra structure on vector bundle $\mathbb{V}(M)$ (~~coalgebra~~ co-bialg and module structure)

$\Big\updownarrow$ bij

$M \otimes M \longleftarrow M$ co-multiplicat

Can give various axioms

To point $\mathbb{V}(M)$ is to give elt. of $\mathrm{Hom}_K(M, K)$.

"unit"

$\mathbb{V}(M)(K)$.

Can globalize to affine scheme over S. ~~[illegible]~~ $\mathbb{V}(M)$

$\underline{M}$ $S' \longrightarrow \Gamma(S', M_{S'})$.

group scheme over K

$$\begin{array}{ccc} G & & \text{bialgebra } A \\ \downarrow & & \uparrow \text{(with axioms)} \\ K & & K \end{array}$$

$\mathbb{V}(M)$. etc.

$K = k$

Examples (a) $G_{a,k}$. represented by $A = K[T]$.

$A \longrightarrow A \otimes A$

$k[T] \longrightarrow k[1 \otimes T, \; T \otimes 1]$

$$\delta(T) = 1 \otimes T + T \otimes 1.$$

Pf Look at def.

$$G \times G \xrightarrow{\pi} G$$

$$A \otimes A \longleftarrow A$$

(i) G_m $\qquad A = k[T, T^{-1}]$ alg. str. den.

$G_m \times G_m \longrightarrow G_m$ $\qquad \delta(T) = (T \otimes 1)(1 \otimes T) = T \otimes T$

if $S = T - 1$. $\qquad \delta(S) = S \otimes 1 + 1 \otimes S + S \otimes S$

(S nicer in sense ~~that~~ it vanishes at 1)

(a) and (b) induce diagonal map for their subgroups.

More on functors.

$$C \hookrightarrow \hat{C}$$

F, G. cont. $(C \to \text{Sets})$

Seek $\underline{Hom}(F, G) \in \hat{C}$. ($Hom(F,G)$ = set of natural transformations)

$$\underline{Hom}(F, G)(S) = Hom(F|S, G|S).$$
$$= Hom_S(F \times S, G \times S)$$
$$= Hom(F \times S, G).$$

$$\| \quad Hom(S, \underline{Hom}(F,G))$$

C/S objects over S

$F|S$ same as fibre product

$$\begin{array}{ccc} F & \leftarrow & F \times S \\ \downarrow & & \downarrow \\ e & \leftarrow & S \end{array}$$

$F(S) = Hom(S, F)$

e final functor — assigns to everything the one point set.

Check that $\underline{Hom}(F, G)$ is functor of S.

$$\boxed{\begin{aligned} Hom(S, \underline{Hom}(F, G)) &\cong Hom(F \times S, G) \\ &\cong Hom_S(F_S, G_S) \end{aligned}}$$

by symmetry $\cong Hom(F, \underline{Hom}(S, G)) \cong Hom_F(S_F, G_F)$.

Same formula holds, even if S not representable functor.

also $\underline{\mathrm{Hom}}(F, \underline{\mathrm{Hom}}(G,H)) \cong \underline{\mathrm{Hom}}(F\times G, H)$

$\cong \underline{\mathrm{Hom}}(G, \underline{\mathrm{Hom}}(F,H))$

proof omitted.

If F, G have composition law.

can talk about $\underline{\mathrm{Hom}}_*(F, G)$ by taking only those Hom's compatible with composition law $*$.

Similarly F, G "ring functors", etc.

elt. of $\mathrm{Hom}(F\times S, G)$ amounts to, $\forall T$, $s \in S(T)$

$s: F(T) \longrightarrow G(T)$

$$\begin{array}{c} F(T)\times S(T) \\ \downarrow \\ G(T) \end{array} \qquad T' \rightleftarrows \begin{array}{c} S(T), s \\ S(T'), s' \end{array} \qquad \begin{array}{ccc} s: F(T) & \longrightarrow & G(T) \\ \downarrow & & \downarrow \\ s': F(T') & \longrightarrow & G(T') \end{array}$$

commutative.

elt. belongs to Hom^* if s respect structure in question.

Ring functor $\mathcal{O}$.

F, G ~~...~~ $\mathcal{O}$-modules.

$$S \times F \longrightarrow G \qquad \in Hom(S \times F, G).$$

instead of S acting on $\mathcal{O}$-module F.

same as. $F|_S \to G|_S$.

G group functor. M module functor over $\mathcal{O}$.

$$G \times M \longrightarrow M.$$

$G(T)$ gives $\forall T$, $\mathcal{O}(T)$ autom. of $M(T)$.

~~...~~ G ~~... endo. of $M|_G$~~.

action is equivalent to automorphism of $M|_G$

interpretation in $Hom_G(M_G, M_G)$. since

$= Hom(G \times M, M)$.

enough to check conditions on $T = G \times G$

~~About~~

$s = pr_1$

$t = pr_2$.

$(s, t) \in G(T)$

$M|_T$.

$$\begin{array}{ccc} T & \longrightarrow & G \times G \\ & \searrow & pr_1 \downarrow \ \downarrow pr_2 \end{array}$$

<u>Apply to</u>

Spec A. = G affine scheme$/K$ ~~with ... ~~

$|$

K

M module over K.

$\underline{M}$ defined.

G acts on M. $\Longleftrightarrow$ endo. of $\underline{M}|G$

$\underline{M \otimes_K A}$

$\updownarrow$

y endo. of $M \otimes_K A$

$\updownarrow$

$M \xrightarrow{u} M \otimes_K A$

K homomorphism (comp. of ~~base~~ extension).

$G \times \underline{M} \to \underline{M}$ the maps which restrict module structure correspond to

[but not just any morphism $G \times \underline{M} \to \underline{M}$ ...]

Associativity $\boxed{g(g'x) = (gg')x.}$

~~constivity~~

$$G \times G \xrightarrow{\pi} G$$

p_1

p_2

$$M \otimes_k A \otimes_k A \longleftarrow M$$

is commutativity of diag corresponding to

$$\begin{array}{ccc} G \times G \times \underline{M} & \longrightarrow & G \times \underline{M} \\ \downarrow & & \downarrow \\ G \times \underline{M} & \longrightarrow & \underline{M} \end{array}$$

action of G on M

Same as endo. of $\underline{M}_G$.

reduced to whether 2 ends in $\underline{A}$ of M_G are same.

$G \times G \xrightarrow[p_1 \ p_2]{\pi_{\#}} G$

$\pi^{\#}(u)$ $p_1^{\#}(u)$ $p_2^{\#}(u)$.

and $\pi^{\#}(u) = p_1^{\#}(x)\, p_2^{\#}(x)$. is given by comultiplication of

$$\begin{array}{ccc} A \otimes A \otimes M & \xleftarrow{id_A \otimes \mu} & A \otimes M \\ \uparrow{\scriptstyle \pi \otimes id_M} & & \uparrow{\scriptstyle \mu} \\ A \otimes M & \xleftarrow{\mu} & M \end{array}$$

map giving comodule structure.

(This is not entirely clear and requires proof)

Cartier Duality

Commutative

G group functor over S.

$$\begin{array}{c} G \\ \downarrow \\ S \end{array}$$

$\underline{\mathrm{Hom}}_{gr}(G, H)$ — H group function also. Hermann its dual (H = G_m sum)

$\underline{\mathrm{Hom}}_{gr}(G,H)_{(S')} = \mathrm{Hom}_{gr}(G_{S'}, H_{S'})$ is group since have only comm. groups here.

$\qquad\qquad\qquad\qquad\qquad\qquad \| $

$\qquad\qquad\qquad\qquad\qquad D(G)(S')$ (H fixed).

Def $D(G) = \underline{\mathrm{Hom}}_{gr}(G, H)$. contravariant in G.

$G \longrightarrow D(G)$

$D(D(G)) \longleftarrow G$ natural homomorphism

" $D(G)$ = dual of G wrt H "

$G \times D(G) \longrightarrow H$.

$\mathrm{Hom}_{gr}(K, \underline{\mathrm{Hom}}(G, H))$

take identity

$= \mathrm{Hom}(K \times G \longrightarrow H)$

additive in G and K.

= bilinear pairing $K \times G \longrightarrow H$.

G reflexive w.r.t. H if this is an isomorphism eg category 1 pt. just groups. $H = \mathbb{Z}$.

$D(G) =$ ordinary dual.

only reflexive ones are free of finite type.

$G_m = H$ ~~need~~ need H-commutative

Cartier dual of G : dual wrt H.

raise to n^{th} powers

G_m $\underline{Hom}(G_m, G_m) \xleftarrow{\cong} \mathbb{Z}_S$

constant ring functor. 'group scheme'

ring functor. : $\exists$

Spec $\mathbb{Z}$.

$S = \operatorname{Spec} \mathbb{Z}$

: for all T

$\underline{Hom}(G_{mT}, G_{mT}) \simeq \mathbb{Z}_T$.

$\Rightarrow$ $Hom(G_{mT}, G_{mT}) \cong \underline{\text{const. maps}}\ (T, \mathbb{Z})$.

$$D(\mathbb{G}_m) = \mathbb{Z}_S.$$

$$D(\mathbb{Z}_S) \cong \mathbb{G}_m.$$

$\therefore$ $\mathbb{G}_m$ and $\mathbb{Z}_S$ are duals of each other

finite product of reflexives is reflexive.

G finite comm. locally free group scheme. $= \mathrm{Spec}(A)$

A sheaf of Algebras. A locally free of finite type.

$\begin{array}{c} G \\ | \\ S \end{array}$

Then so is $D(G)$.

and G is reflexive.

$H = \mathbb{G}_m$

exercise!

~~Wed Feb 17~~

Friday (10:30 here)

i.e. p94. unitary action.

$$\begin{array}{ccc} A & & G \\ \Big\downarrow \varepsilon & & \Big\uparrow \varepsilon \\ K & & S = \operatorname{Spec} K \end{array}$$

~~want to see the how~~ see how ε acts on M.

$$\begin{tikzcd} M \arrow[r, "u"', "\text{comodule structure}"] & M \otimes A \arrow[d, "id_M \otimes \varepsilon"] \\ & M \end{tikzcd}$$

1

ε: unit section

In these diagrams (p94, 95) have only used comultiplication of A, not the algebra structure of A.

A has coalgebra structure + K module $\iff$ $V(A)$ has algebra structure.

M is comodule over A $\iff$ $\underline{M}$ is module over $V(A)$ (extending structure of $\underline{M}$ as module over O_K)

Example Γ commutative group (just abstract).

S arbitrary base

constant group scheme over S.

$$D(\Gamma_S) = \underline{\mathrm{Hom}}_{gr}(\Gamma_S, \mathbb{G}_{m_S})$$

$$\parallel$$

$$(D(\Gamma_{\mathbb{Z}}))_S.$$

gr = group.

claim $D(\Gamma)$ is representable.

$$\parallel$$

$$D(\Gamma_{\mathbb{Z}})$$

$$D(\Gamma)(T) = \mathrm{Hom}_{\substack{\text{group}\\ \text{schemes}}}(\Gamma_T, \mathbb{G}_{m_T})$$

$$= \mathrm{Hom}_{\substack{\text{abstract}\\ \text{group}}}\Big(\Gamma, \underbrace{\Gamma(T, \mathbb{G}_{m_T})}_{\Gamma(T,\mathcal{O}_T)^*}\Big)$$

$\mathrm{Hom}_T(T, \mathbb{G}_{m_T})$ = def.

$$\mathrm{Hom}_{\substack{\mathbb{Z}\text{-algebras}\\ = \text{rings}}}(\mathbb{Z}[\Gamma], \Gamma(T, \mathcal{O}_T))$$

Thus $D(\Gamma) = \operatorname{Spec} \mathbb{Z}[\Gamma]$. (affine group scheme).

$\boxed{\text{diagonal group of type } \Gamma}$

eg $D(\mathbb{Z}) \cong G_m$

$D(\mathbb{Z}^r) \cong (G_m)^r$

$\mathbb{Z} \xrightarrow{n} \mathbb{Z} \to \mathbb{Z}/n\mathbb{Z} \to 0$

$D(\mathbb{Z}/n\mathbb{Z}) \cong \mu_n$.

$M' \to M \to M'' \to 0$

$0 \to D(M'') \to D(M) \to D(M')$

) additive functor of Γ.

∴ know for every finite group.

$\mathcal{C} \subset \hat{\mathcal{C}}$

Notation

$$\begin{array}{c} F \\ \downarrow \\ T \end{array} \qquad \Gamma(T,F) = \operatorname{Hom}_T(T,F)$$

"sections".

Comultiplication $\mathbb{Z}[\Gamma] \xrightarrow{\delta} \mathbb{Z}[\Gamma \times \Gamma]$

$\gamma \longmapsto (\gamma,\gamma)$.

augmentation $\mathbb{Z}[\Gamma] \longrightarrow \mathbb{Z}$

$\gamma \longmapsto 1$

$$D(\Gamma) \times \Gamma_2 \longrightarrow G_m$$

K any ring

$$\operatorname{Spec} K[\Gamma] = D(\Gamma_K)$$

What does it mean for $D(\Gamma_K)$ to act on M?

$$M \xrightarrow{\pi} M \otimes k[\Gamma] = M^{(\Gamma)\,(\mathrm{sum})} \xrightarrow{P_\gamma} M$$

π determined by $\{P_\gamma\}_{\gamma \in \Gamma} = \{\pi_i\}$

$P_\gamma : M \longrightarrow M$. $\{\forall x$, all but a finite no. of $P_\gamma(x)$ are zero$\}$

$A \otimes A = k[\Gamma \times \Gamma]$ $\quad \sum_{i,j} (\pi_i(x), i, i)$

Associativity

$$\begin{array}{ccc} M^{(\Gamma \times \Gamma)} & \xleftarrow{id_M \otimes \delta} & M^{(\Gamma)} = M \otimes A \quad (\pi_i(x), i)_{i \in \Gamma} \\ \uparrow \pi \otimes id\, A & & \uparrow \\ M \otimes A & \longleftarrow & M \ni x \end{array}$$

$\sum_i \sum_j (\pi_j(\pi_i(x)), j, i)$

$M \otimes A = M^{(\Gamma)}$

$\sum_i (\pi_i(x), i)$

Thus $\pi_j(\pi_i(x)) = 0$ if $i \neq j$

$\pi_i(\pi_i(x)) = \pi_i(x)$

$$\boxed{\begin{array}{ll} \pi_j \pi_i = 0 & i \neq j \\ \pi_i^2 = \pi_i & \end{array}}$$

Is id by identity

$$M \xrightarrow{\pi} M^{(\Gamma)} \xrightarrow{id_M \otimes \varepsilon_i} M$$

$$x \longrightarrow \sum(\pi_i(x), i) \longrightarrow \sum \pi_i(x).$$

$$\boxed{id_M = \sum \pi_i}$$

since all but finite number are $\neq 0$.

ie $$M = \bigoplus_{\gamma \in \Gamma} M_\gamma$$

grading of M of type Γ.

$$M_\gamma = \text{image } \pi_\gamma.$$

Above is functorial

$$M = \bigoplus M_\gamma$$

$$N = \bigoplus N_\gamma.$$

$M \to N$ commutes with action of ~~diagonal~~ group

$\Longrightarrow$ maps $M_\gamma \to N_\gamma$.

$x \in M$ $\quad x \in M_\gamma$ $\quad \gamma : G_x \xrightarrow{\chi_\gamma} \mathbb{G}_m$ character.

$\Leftrightarrow \quad gx = \gamma(g)\, x$

for all $g \in G(k')$?

$G = D(\Gamma)$?

Category semi-simple ~~if~~ if k field.

G affine group scheme

G^0 - connected component
- finite type over k

Then (i) G/G^0 is of order prime to p.
constant g. scheme [quotients yet to be defined]
[need not be commutative]

k $\quad$ char $k = p > 0$
(alg closed)

(ii) G^0 is diagonal.
$\cong \prod \mathbb{G}_m \prod \mu_{p^r}$

due to (Nagata ?)

$x \longmapsto x^p$ in sym. alg is additive.

$$Gl(n) \qquad = \mathrm{Sym}^p(E) \longleftarrow E^{(p)}$$

subspace gen. by x^p's.

char p)

E indiv'l vs standard rep.

diagonal groups very rigid.

$D(\Gamma)$ is reflexive group, ~~scheme~~

(~~equiv~~ $\Gamma_{\mathbb{Z}}$ is reflexive.).

constant group scheme.

equiv $\Gamma_{\mathbb{Z}} \xrightarrow[\text{nat. map}]{\approx} D(G)$

$D(G) = D(\Gamma)$.

$S = \coprod_\gamma S_\gamma$ $G_S \longrightarrow G_{m_S}$

$S = \operatorname{Spec} k$

$S = \operatorname{Spec} k$ (assume).

$\chi_\gamma : G \to G_m$.

$M = k$.

free module over k.

how can G operate on module of rank 1

$\Updownarrow$

decomposition of k

but such ~~correspond~~ ~~can't decompose~~ to decomp. of spectrum of k this gives reflexivity.

$$G = \mathrm{Spec}(\mathcal{A})$$

$$\Big|$$

$$S$$

$\mathcal{A}$ sheaf of algebras over $\mathcal{O}_S$ locally free of finite type as sheaf of modules.

– also commutative group scheme.

– claim – is reflexive. $D(G)$ also finite, locally free $/S$.

$$G^* = D(G).$$

$$G^*(S) = \mathrm{Hom}_{\text{group}}(G, G_m)$$

$$= \mathrm{Hom}_{\text{bialgebras (mapping units} \to \text{units)}}(\mathcal{O}_S[T, T^{-1}], \mathcal{A})$$

$$\cap$$

$$\mathrm{Hom}_{\text{algebras}}(\mathcal{O}_S[T, T^{-1}], \mathcal{A})$$

$$\| \quad \Gamma(S, \mathcal{A})^* \subset \Gamma(S, \mathcal{A})$$

$$= \underline{A}(S).$$

$\therefore\ G^\times \subset \underline{A} = \mathbb{V}(A^\vee)$ (for all G affine $/S$)

since A locally free finite type / $\underline{A}$ = "group ring" of G"

$$G^\times \subset \underline{A}^\times \text{ even...}$$

Now ~~take~~ only alg. homs compatible with diagonal map.

$\delta(T) = T \otimes T.$ —— $\mathrm{Hom}_{alg}(\mathcal{O}_S(T, T^{-1}), A).$

$g \in \Gamma(S, A)^\times$ $\boxed{\delta(g) = g \otimes g}$ δ and ε for G.

$\varepsilon(T) = 1.$ $\boxed{\varepsilon(g) = 1}$

$\boxed{g \text{ invertible}}$ must check is consequence of those two.

$A^\vee \longrightarrow \mathcal{O}_S$ $A^\vee$ is commutative unitary algebra.

$$G^* = \mathrm{Spec}(A^\vee)$$

algebra by transposing co-algebra structure on A.

$\exists$ diagonal map $A^\vee \longrightarrow A^\vee \otimes A^\vee$.

transpose of $A \otimes A \to A$.

Friday Feb. 19.

~~A coalgebra~~

Question of Drinfeld — what does it mean for $M \to M$ to be linear?

$\mathcal{C}$ category $\qquad \mathcal{O}$ = ring functor (contravariant) (the exact scheme ... global sections usual ring functor)

$$\begin{array}{ccc} \underline{N} & \xrightarrow{f} & \underline{M} \\ \downarrow & & \downarrow \\ X & \longrightarrow & Y \end{array}$$

$\underline{N} \xrightarrow{f} \underline{M}$ modules over $\mathcal{O}$.

$X \longrightarrow Y$ objects in category

What does it mean for f to be linear?

f same as g: $N \xrightarrow{g} M_X$, X

new scheme?

$X = \operatorname{Spec} B$ $\qquad$ $Y = \operatorname{Spec} A$

$M_B = M \otimes_A B.$

Translation

$$\begin{array}{ccc} N & & M \\ | & & | \\ B & \longleftarrow & A \end{array}$$

$\left(\underline{M}(C) = M \otimes_A C \quad C:\ A\text{-algebra} \right)$

M: A-module.
N: B-module.

giving f amounts to linear map $N \longrightarrow M_B$.

Now say $N = M_B$. Seek B linear maps $M_B \to M_B$.

same as A linear maps $M \longrightarrow M_B$

$\underline{N} = \underline{M}_X$

$$(g, g', x) \rightrightarrows \begin{matrix}(gg', x)\\ (g, g'x)\end{matrix}$$

$$\underline{M}_{G \times G} \longrightarrow \underline{M}_G$$

Our case

$$\begin{array}{ccccc} G \times G & & G & & \\ M_{A \otimes A} & & M_A & \xleftarrow{K\text{-linear map}} & M \\ A \otimes A & \longleftarrow & A & \longleftarrow & K \end{array}$$

+ linear w.r.t. some ideals.

$x \otimes 1 \xleftarrow{\Delta} x$

$x \otimes x \xleftarrow{\varepsilon} x$

$$G \times G \times M \xrightarrow{\text{compatible with } G \times G \xrightarrow{\text{mult.}} G} G \times M$$

$$(g, g', x) \longmapsto (gg', x)$$

compatible with 1st proj.

$$G \times M \longrightarrow M$$

$$(g, gx)$$

$A \xrightarrow{\delta} A\otimes A$ co-mult.

$A^\vee$ has algebra structure.

$$\begin{tikzcd} A & A^\vee \otimes A^\vee \arrow[l, "\delta^*\,(\mathrm{def.})"'] \arrow[d] \\ & (A\otimes A)^\vee \arrow[ul, "t_\delta"] \end{tikzcd}$$

$(u, v) \mapsto u\otimes v$

K

$A^\vee$: pts of v.b. with values in K itself.
$\mathbb{V}(A)(K) = A^\vee$.

δ commutative $\longleftrightarrow$ $\mathbb{V}(A)$ has comm. alg. str.

$\Downarrow$

$A^\vee$ is commutative

Same for associative and unitary properties.

can go backwards?

$A^\vee$ ass $\Rightarrow A$ ass?

$A\otimes A \xrightarrow{\pi} A$ algebra structure

$A\otimes A \xrightarrow{\pi} A$

$(A\otimes A)^\vee \xleftarrow{{}^t\pi} A^\vee$

$\uparrow$ wrong way. But is iso. is A is proj of finite type.

$A^\vee \otimes A^\vee$

Then get comultiplication on $A^\vee$. Otherwise use completed tensor product,

$(A \otimes A)^\vee = A^\vee \hat{\otimes} \hat{A}$?

A locally free

Then to give comult. on $A^\vee$ is same as to give alg. structure on A.

π is ass, comm. unitary

$\iff \pi^t$ is . . . .

locally free

A ~~proj.~~ finite type.

A is bialgebra

$A \xrightarrow{\delta} A \otimes A$

δ hom. for algebra structure.

[autodual — same as to say i.e. $A \otimes A \xrightarrow{\pi} A$ is compatible with δ]

$A^\vee$ with dual operations.

— these operations are compatible and make $A^\vee$ into a bialgebra

A comm. ass. unitary for comult $\iff$ $A^\vee$ comm. ass. unitary for mult,

mult $\longleftrightarrow$ comult.

Antiequivalence in category of bialgebras $A \longrightarrow A^\vee$.

$u : A \longrightarrow A$
module endo.

u is antipodism $\iff$ $u^\vee$ is antipodism
(autodual notion).

$u^\vee \quad A^\vee \longleftarrow A^\vee$

<u>Def of antipodism</u> (acts like inverse)

$g \mapsto (g,g) \mapsto (g^{-1}, g) \mapsto (g^{-1}g)$

$$\begin{tikzcd} G \arrow[r, "\delta_G"] \arrow[drr, "e"'] & G \times G \arrow[r, "S \times id_G"] & G \times G \arrow[r, "\pi_G"] & G \\ & & S \arrow[ur, "\varepsilon"'] & \end{tikzcd}$$

left ~~unit~~ inverse

condition on bialgebra.

$$\begin{tikzcd} A & A \otimes A \arrow[l, "\pi"'] & A \otimes A \arrow[l, "S \otimes id"'] & A \arrow[l, "\delta"'] \arrow[dl, "\varepsilon"] \\ & & k \arrow[ull, "e"] & \end{tikzcd}$$

~~reversing arrows gives same diagram.~~
∴ notion is autodual.

autom. for module str.

Antipodism = antiautom. of algebra structure, of ~~coalgebra~~ mult structure
and makes above diagram commutative [assume ∃ unit, counit]

$G \longmapsto G^*$

$G = \mathrm{Spec}(A)$ $\quad$ $G^* = \mathrm{Spec}(A^\vee)$.

$\boxed{A \text{ locally free finite type}}$

$S = \mathrm{Spec}\, K$. commutative over S. $\quad$ so is G^*

G is group scheme (3 axioms) $\Longleftrightarrow$ G^* is group scheme

$\underline{\mathrm{Hom}}_{\text{monoid}}(G, G_m) \hookrightarrow G^*$

[group functor]

$(G^*)^\circ$ $\left[\text{subgroup of invertible elements}\right]$ $\Rightarrow (G^*)^\circ$ representable by open subset of G^*. $\subset G^*$.

$\mathrm{Hom}(G_T, G_{m_T})$ = group of elements of $G^*(T)$ (monoid) which are invertible.

Assume $T = S$ by making base change

Hs $G \longrightarrow G_m$ correspond to $A^\vee \xrightarrow{g} K$ which are invertible

$g \in A$

$$\left.\begin{aligned} \delta g &= g \otimes g \\ \varepsilon(g) &= 1 \end{aligned}\right\} \text{ group like elts. of } A.$$

Similarly $\mathfrak{I}$ make monoid, with operation induced by alg. str. of A.

(if $\mathfrak{I}'$ has above properties so does gg'.)

To be invertible in A or $G^*(K)$ same.

ie g' is group like element too.

If G is group, G^* is group.

$$(G^*) = (G^\vee)^\circ$$

and $G^* = \underline{\mathrm{Hom}}_{gr.}(G, \mathbb{G}_m)$ as desired.

"Cartier dual".

(category of locally free finite type comm. group schemes over S)

Examples of Cartier Duality

$$\mathbb{Z}/n\mathbb{Z} \longleftrightarrow \mu_n \longrightarrow \text{(almost def)}$$

char p $\quad \alpha_p$ $\longleftrightarrow$ α_p

and f +

$D(\alpha_{p^n}) \neq \alpha_{p^n}$ if $n > 1$

α_p

$\|$

$$\mathrm{Spec}\Big(\overbrace{k[T]/T^p}^{A}\Big)$$

alg. str = mag mult.

$\delta T = T \otimes 1 + 1 \otimes T$

basis $\quad x_i = T^i \quad 0 \le i \le p-1$

table $\quad x_i \cdot x_j = \begin{cases} x_{i+j} & \text{if } i+j < p \\ 0 & \text{if } i+j \ge p. \end{cases}$

$$\delta x_i = \sum_{j+k=i} \frac{(j+k)!}{j!\,k!} \cancel{T^j \otimes T^k}\, x_j \otimes x_k.$$

unit, counit

Take transpose

$$x_i' x_j' = \langle x_i' x_j', x_k \rangle = \langle x_i' \otimes x_j', \delta x_k \rangle$$

$$= \begin{cases} 0 & \text{if } i+j \neq k. \\ \dfrac{(i+j)!}{i!\,j!} & \text{if } i+j = k. \end{cases}$$

$\therefore\; x_i' x_j' = \dfrac{(i+j)!}{i!\,j!} x_{i+j}'$ if $i+j < p$ (otherwise set zero).

"multiplication with divided powers"

can change basis $Y_i = i!\, x_i'$ $\qquad 0 \le i \le p-1$

$\boxed{i! \text{ all invertible since } = p}$

relation becomes

$$Y_i Y_j = Y_{i+j}$$

$$k[S]/(S^p) \qquad S = Y_1 = X_1'.$$

$$\langle \delta X_1', \; x_i \otimes x_j \rangle = \langle X_1', \underbrace{x_i x_j}_{} \rangle$$

~~x_{i+j} if $i+j < p$~~

$$\begin{cases} 0 \text{ if } (i,j) \neq (0,1),\ (1,0) \\ 1 \text{ otherwise.} \end{cases}$$

$$\therefore \delta X_1' = X_1' \otimes 1 + 1 \otimes X_1'.$$

$$\therefore \delta S = S \otimes 1 + 1 \otimes S.$$

$\therefore$ get α_p.

$D({}_nW_m) \cong {}_mW_n$. Witt groups.
(are group schemes).

$D({}_nW_1) \cong {}_1W_n$

α_{p^n}

to understand must know Witt groups

$\boxed{\text{may do later}}$

Gabriel Demazure complete proof of Dieudonné's thm

Feb 24

Flatness

A ring M flat $\_\otimes M$ exact.

Manin Commutative formal groups: Dieudonné modules?

Dieudonné's papers unreadable

Cartier 2 notes simplify complete readers the Dieudonné theory for comm. rings no proofs.

eg ① $M = A$ is flat. $\otimes A$ = identity functor

② direct sum of flat modules is ~~exact~~ flat.
free modules are flat.

③ a direct factor of a flat module is flat $\Rightarrow$ projective modules are flat

④ M flat over A, $A \to B$ then M_B is flat over B.

(5) $0 \to M' \to M \to M'' \to 0$ exact.

then M', M'' flat $\Rightarrow$ M is flat.

M, M'' flat $\Rightarrow$ M' is flat.

$0 \to M' \to M \to M'' \to 0$ exact.

(6) $\forall N$, then $0 \to N \otimes M' \to N \otimes M \to N \otimes M'' \to 0$ exact

if M'' flat.

(long exact sequence of Tor's).

(7) M flat $\Leftrightarrow$ $\forall \mathfrak{p}$, $M_\mathfrak{p}$ over $A_\mathfrak{p}$ is flat

$\therefore$ is ~~flat~~ local property.

1st $(M \otimes N)_\mathfrak{p} = M_\mathfrak{p} \otimes_{A_\mathfrak{p}} N_\mathfrak{p}$

or since can check exactness at stalks.

$M_\mathfrak{p} = M \otimes_A A_\mathfrak{p}$

M flat $\Rightarrow M_\mathfrak{p}$ flat.

$\Leftarrow$ $\forall$ max ideals, $M_\mathfrak{p}$ over $A_\mathfrak{p}$ is flat.

(transitivity of localization)

[Projectivity is subtler notion]

projective modules rare if not of finite type.

flatness more impt. in alg. geo.

(8) M is projective / finite type $\Leftrightarrow$ M flat of finite presentation.

assumptions on A?

⑨ If A is Artinian local ring, ~~[illegible]~~.

M is flat $/A \iff M$ is free $\iff M$ is projective.

⑩ A Artin. Then M flat $\iff M$ projective.

Faithfully flat M/A

⇕

① -flat

② (a) if sequence becomes exact, it was already exact.

(b) [illegible] $M \otimes N = 0 \Rightarrow N = 0$.

2(a) ⇒ 2(b) clear.

2(b) ⇒ 2(a) Assume M flat and $M \otimes N = 0 \Rightarrow N = 0$.

2(a)

Then $u : N \to N'$, $u = 0 \iff u \otimes id_M = 0$

Meaning of word flat

⑪ A reduced. M finite presentation.

Then M flat $\iff \mathfrak{p} \longmapsto \text{rank}_{\kappa(\mathfrak{p})} M \otimes \kappa(\mathfrak{p})$ ✓

is locally constant.

$\kappa(\mathfrak{p})$ = residue class field

$\mathfrak{p} \subset \mathfrak{q}$

$\frac{1}{\{x\}}$? y

rank $M(\mathfrak{p}) \leq \text{rk} M(\mathfrak{q})$.

not

(12) A local, M finite presentation.

M flat $\Leftrightarrow$ M free $\Leftrightarrow$ M projective.

(13) A anything. M finite presentation.

M free at $\mathfrak{p}$ $\Rightarrow$ M free in nbhd of $\mathfrak{p}$.

(14) $A \supset I$ $\quad$ I nil~~potent~~

$A_0 = A/I$ $\qquad$ $M_0 = M \otimes_A A_0 = M/IM$.

M flat$/_A$ $\Rightarrow$ M_0 is flat $/A_0$ $\qquad$ (base change)

$$gr_I^n M := I^n M / I^{n+1} M \longleftarrow M \otimes (I^n/I^{n+1})$$
$$\qquad\qquad\qquad\qquad\qquad\qquad \parallel$$
$$\qquad\qquad\qquad\qquad\qquad M_0 \otimes_{A_0} I^n/I^{n+1}$$

always onto.

M flat $\Rightarrow$ above hom. is isomorphism.

$\therefore$ get $\quad$ (I nilpotent)

$$M \text{ flat } /A \Leftrightarrow \begin{cases} (1) \; M_0 \text{ flat } /A_0 \\ (2) \; gr^n(M) = I^n M / I^{n+1} M \xleftarrow{\sim} M_0 \otimes_{A_0} I^n/I^{n+1} \quad \forall n \end{cases}$$

"evenly layered".

(if A_0 = field (1) not needed.)

Faithful Flat

0 is flat but not faithfully flat

1) A is faithfully flat $/A$.

2) $0 \to M' \to M \to M'' \to 0$ ~~M flat~~ M', M'' flat.

either M' or M'' faithfully flat.

$\Rightarrow M$ is faithfully flat.

eg $M_1 \oplus M_2$ faithfully flat if M_1 flat, M_2 faithfully flat

3).
$$\begin{tikzcd} M \arrow[r, dash] \arrow[d, dash, "\text{f.f.}"'] & M_B \arrow[d, dash] \\ A \arrow[r, dash] & B \end{tikzcd}$$

M f.f. $/A$ $\Rightarrow M_B$ faithfully flat $/B$

4.) ~~M finite presentation~~

~~M flat~~ M f.f. $\Longleftrightarrow \forall \mathfrak{p} \in \operatorname{Spec} A$ $M_{\mathfrak{p}}/A_{\mathfrak{p}}$ f.f.

M finite presentation. $\Longleftrightarrow$ $M_{\mathfrak{p}}$ flat $(\neq 0)$.

$(\Leftarrow)$ rank everywhere positive

ie is local property

L free $\supset \underset{\neq 0}{m^n L} \cong L_0 \otimes_K m^n$. $\supset K \neq 0$. any subvector space.

A local artin ring. m. $(\neq 0)$

$A/m = K$

m nilpotent.

$m^n \neq 0 \qquad m^{n+1} = 0$.

L/R is not flat

$$M_{n-1} = M \otimes_A A/m^n \text{ is free over } A/m^n$$

but is not free over A itself.

~~such~~ modules free over A/m^n but not over A arise in this manner.

Also A/m is not flat over A.

⑤ A integral domain.

K = field of fractions.

Then K is not faithfully flat (unless $K = A$).

eg $\mathbb{Z}/2\mathbb{Z} \otimes_{\mathbb{Z}} \mathbb{Q} = 0$.

Algebras over A.

$$\begin{array}{ccc} B \text{ algebra}/A & & X \\ \uparrow & & \downarrow f \\ A \text{ ring} & & Y \end{array}$$

B flat if flat as a module.

Flatness local upstairs as well as down.

ie B flat over A $\iff$ $\forall \, \mathfrak{p} \in X$ $\quad (\mathfrak{q} = f(\mathfrak{p}))$

Then $\mathcal{O}_{X,x} = B_{\mathfrak{p}}$ is flat over $\mathcal{O}_{Y,y} = A_{\mathfrak{q}}$.

Proof $N \longrightarrow N \otimes_A B$ exact in M?

base extension B modules

$0 \to N' \to N$ exact $\Rightarrow$ $0 \to N' \otimes_A B \to N \otimes_A B$ exact

of to check at primes of X.

[localization never f.f. except in ~~the~~ if nothing thrown away]

$A \longrightarrow B$ faithfully flat means:

$$① \; B \text{ flat } ② \begin{cases} (a) \; A \to B \text{ injective and } B/A \text{ flat.} \\ \qquad \Updownarrow \text{ assuming } ①. \\ (b) \; \operatorname{Spec} B \longrightarrow \operatorname{Spec} A \text{ surjective} \end{cases}$$

NB $\forall \mathfrak{p} \in A$, $B \otimes_A k(\mathfrak{p})$

is enuf to be onto Max A.

$F(\mathfrak{p})$ = residue field at $\mathfrak{p}$

$$\begin{tikzcd} B \\ A \arrow[u] \end{tikzcd}$$ local hom. of local rings. B flat$/A$ $\Rightarrow$ B faithfully flat over A.

Now assume
$$\begin{tikzcd} \mathcal{O}_{X,x} & \\ X_y \arrow[u] & X \arrow[d] \\ & y \in Y \end{tikzcd}$$

y max'l. $\Rightarrow X_y \neq \emptyset$.

Seek $X_z \neq \emptyset$ if z arbitrary.

Choose $y \in \overline{\{z\}}$ y max'l. $X_y \neq \emptyset$

$$\begin{tikzcd} \operatorname{Spec} \mathcal{O}_{X,x} \arrow[r] \arrow[d] & X \arrow[d] \\ z \in \operatorname{Spec} \mathcal{O}_{Y,y} \arrow[r] & Y \end{tikzcd}$$

$\therefore$ z comes from $\operatorname{Spec} \mathcal{O}_{X,x}$. Hence some elt of X.

Extend to ~~rings~~ ~~spaces~~ schemes + morphisms:

$$\begin{tikzcd} X \arrow[d, "f"] \\ Y \end{tikzcd}$$

f flat if $\left\{ \begin{array}{l} F \to f^*(F) \\ \mathrm{Mod}(Y) \to \mathrm{Mod}(X) \text{ is exact.} \\ \Updownarrow \\ \forall x \in X,\ \mathcal{O}_{X,x} / \mathcal{O}_{Y,f(x)} \text{ is flat.} \end{array} \right.$

If $Y = \operatorname{Spec} A$, $X = \operatorname{Spec} B$

then get simply B is flat over A.

Y scheme. $\mathcal{M}$ sheaf of $\mathcal{O}_Y$-modules.

Then $\mathcal{M}$ flat if either (a) $\mathcal{N} \longrightarrow \mathcal{M} \otimes \mathcal{N}$ exact
$\operatorname{Mod}(Y) \longrightarrow \operatorname{Mod} Y$.

(b) $\forall \wp \in Y$, $\mathcal{M}_\wp / \mathcal{O}_{Y,\wp}$ is flat.

if $Y = \operatorname{Spec} A$, $\mathcal{M} = \tilde{M}$ then get usual

Faithfully flat definition.

X

$\downarrow f$

Y

$\mathcal{M}$ f.f. is (a) $\mathcal{N} \to \mathcal{M} \otimes \mathcal{N}$ is flat and faithful

f is f.f. $\Leftrightarrow$ $\left\{ \begin{array}{l} \left. \begin{array}{c} N \to f^*(N) \\ \operatorname{Mod} Y \to \operatorname{Mod} X \end{array} \right\} \text{ is both exact and faithful.} \\ \Updownarrow \\ f \text{ flat and surjective.} \end{array} \right.$

11 AM.

Talk Tomorrow. 1:30 social issues

~~4:00 other talk~~

Feb 25 A local artin ring $K = A/m$

E flat over A, Prove E free.

$E_0 = E \otimes_A K$ free over K

$\exists L$ free over A such $L \otimes_A K \cong E \otimes_A K$.

$$\begin{tikzcd} L \arrow[dr] \arrow[d, "l"', "\text{lift}"] & \\ E \arrow[r] & E_0 \end{tikzcd}$$

enuf to show l iso:

$A \supset m_{\text{nilpotent}}$ l iso since $L_0 \to E_0$ iso.

pf

$$\begin{tikzcd} \operatorname{gr} L \arrow[r, "\text{iso}"] & \operatorname{gr} E \\ \operatorname{gr} A \otimes_{A_0} L_0 \arrow[u, "s"] \arrow[r, "\approx"'] & \operatorname{gr} A \otimes_{A_0} E_0 \arrow[u, "s"'] \end{tikzcd}$$

since L and E flat.

since $L_0 \to E_0$ iso.

$\therefore\ L \longrightarrow E$ isomorphism itself. [well known!]

Étale morphisms

Topological analogy.

$$\begin{array}{c} X \\ \downarrow f \\ Y \end{array}$$

f étale $\Longleftrightarrow$ local homeomorphism. $\forall x \in X, \exists\, \mathcal{U} \ni x$ open such that $f|\mathcal{U}$ is a homeomorphism of $\mathcal{U} \xrightarrow{\sim} f(\mathcal{U})$. [$f(\mathcal{U})$ open]

[sheaves = étale spaces over Y]

$$\begin{array}{c} X \\ \downarrow f \\ Y \end{array}$$

étale covering $\Longleftrightarrow$ fiber bundle with discrete fibers.

$\forall y \in Y, \exists\, \mathcal{U} \ni y$ open such that $f^{-1}(\mathcal{U}) \simeq I \times \mathcal{U}$

$$\begin{tikzcd} f^{-1}(\mathcal{U}) \arrow[r, "\simeq"] \arrow[dr] & I \times \mathcal{U} \arrow[d, "pr_2"] \\ & \mathcal{U} \end{tikzcd}$$

I a discrete topological space [I could be empty]

$\mathrm{Cov}(Y)$ = category of étale coverings.

If $Y = Y' \amalg Y''$, then $\mathrm{Cov}(Y) = \mathrm{Cov}(Y') \times \mathrm{Cov}(Y'')$.

(or any family of (clopen) components)

Assume Y connected, locally connected,

every pt has fundamental system of connected nbhds.

locally simply connected.

every pt has ~~[illegible]~~ fundamental system of simply connected spaces.

simply connected $\Leftrightarrow$ all étale coverings are product with discrete space

$y \in Y$.

$$\mathrm{Cov}\, Y \xrightarrow{F_y} (\mathrm{Sets})$$

$$X \longrightarrow X_y \quad \text{fibre at } y.$$

$$\pi \underset{\text{def}}{=} \mathrm{Aut}(F_y) = \pi_1(Y, y), \quad \boxed{\text{fundamental group}}$$

Actually $\mathrm{cov}\, Y \xrightarrow{F_y} (\pi\text{-sets})$

Thm 1 F_y is an equivalence of categories (with assumptions above on Y).

Thm 2 If $y, y' \in Y$ $F_y \simeq F_{y'}$. not canonically.

compatible with (infinite) inverse limits etc.

Thus can talk about group covering, ring covering etc.
[illegible]
group on which π acts.

get at π in usual way with loops. $F_y : Cov(Y) \xrightarrow{\simeq} \pi\text{-Sets}$.

Given loop $\in \pi_1(Y, y)$ acts on functor F_y as follows

lift γ to X "unique" $X_y \xrightarrow{\gamma} X_y$

fibre product etc.

$$\begin{array}{ccc} X_I & \longrightarrow & X \\ \downarrow & & \downarrow \\ I & \longrightarrow & Y \end{array}$$

$X_I = E \times I \cong (X_I)_0 \times I$ gives iso. $X_y \cong X_z$ depends only on homotopy class of the path.

Might not have any loops except constant loop.

Any connected component of étale covering is again étale covering.

X

~~Covering connected~~ $\Leftrightarrow$ action of π is transitive

$\boxed{\text{SGA 1} \quad \underline{V} \quad \text{adaptation of above to algebraic geometry}}$

To copy directly into alg. geo. would get too restrictive a notion.

To get idea, look over $\mathbb{C}$ and pass to analytic spaces.
Want algebraic notion to agree with this.

$$\begin{array}{ccc} \mathcal{O}_{X,x} & x & X \\ \uparrow & & \downarrow \\ \mathcal{O}_{Y,y} & y \in & Y \end{array}$$

iso of rings in analytic case if covering étale

alg local rings and anal. local rings have same completion

x, y closed.

to say here too restrictive, have

look at completion

$\mathbb{C}$ étale $\Leftrightarrow$ $\begin{array}{c}\mathcal{O}_{X,x} \\ \uparrow\cong \\ \mathcal{O}_{Y,y}\end{array}$ $\Leftrightarrow$ $\mathcal{O}_{X,x}$ flat over $\mathcal{O}_{Y,y}$ and $\dfrac{\mathcal{O}_{X,x}}{m_y \mathcal{O}_{X,x}} = k$.

(schemes) $f : X \longrightarrow Y$ étale

Definition 1 f is locally of finite presentation.

locally $B \leftarrow A$ finite no. of generators and relations, as algebras.

f flat

$\forall y \in Y$, $X_y \simeq \coprod_{\text{(finite or infinite)}} \operatorname{Spec} K_i'$

$k(y) =$ r.c.f. of Y.

$$\begin{array}{ccc} X_y & \longrightarrow & X \\ \downarrow & & \downarrow \\ \operatorname{Spec} k(y) & \longrightarrow & Y \end{array}$$

$K_i'/k(y)$ = finite separable alg. extension.

$\Updownarrow$

$X_{\bar{y}} \cong \coprod \operatorname{Spec} \overline{k(y)}$

$\cancel{\text{Spec } k(y)}$ $X_{\bar{y}}$ = fibre over $\operatorname{Spec} \overline{k(y)}$.

Definition 2 $X \xrightarrow{f} Y$ $\forall x \in X$

$y = f(x)$

$\exists\, U \ni y$ affine open nbhd. and U'

$\operatorname{Spec} A = U \longleftarrow U' = \operatorname{Spec} A'$ affine

f. flat over U.

or that

[can even take base change étale]

after base change

$$\begin{tikzcd} X \arrow[d] & X' \arrow[l, hook'] \arrow[d, "f'"] \\ U & U' \arrow[l, hook'] \end{tikzcd}$$

$y \leftarrow$ $\in y'$

$\exists$ $x' \in X'$ nbhd called V' such that $f' : V' \to U'$

is open immersion.

[handy version]

[closest to intuitive notion]

reference SGA I exp I.

1 bis Y locally noetherian.

- f locally of finite type. (a ring) [frees of finite type]
- $\forall x \in X$, $y \in f(x)$ $B = \widehat{\mathcal{O}}_{X,x}$ flat, finite

$\wedge$ wrt max ideal.

$$A = \widehat{\mathcal{O}}_{Y,y} \to B$$

$k = A/m$ $\quad B \otimes_A k$ = finite separable extension of k.

(if k alg closed, will be just k itself).

Definition 3 assume f locally of finite presentation

$$\begin{tikzcd} X \arrow[d, "f"] & Y'_0 = V(I) \arrow[l, "\text{given}"'] \arrow[d] \\ Y & Y' = \operatorname{Spec} B \arrow[l] \arrow[ul, dashed, "\exists \text{ unique lifting}"] \end{tikzcd}$$

I nilpotent ideal.

$\forall\, Y'$ affine, then

$$X(Y') \xrightarrow{\sim} X(Y'_0)$$

(if don't require unique lifting get smooth).

for étale can do nil ideal.
sufficient to have $I^2 = 0$.

Example 1

$$\begin{tikzcd} \cdot \arrow[d] & k_1 \oplus k_2 \oplus \cdots \oplus k_r \\ \operatorname{Spec} K & k \arrow[u] \end{tikzcd}$$

finite direct sum of finite separable extensions

Eg 2

$$\begin{tikzcd} X \arrow[d] \\ Y = \operatorname{Spec} A \end{tikzcd}$$

$\begin{cases} A \text{ local artin ring} \\ k(A) = A/m \text{ separably closed} \end{cases}$

Then X étale$/Y$: direct sum of copies of Y.

$$\begin{array}{ccc} X' & & X_0 \\ \downarrow{\scriptstyle \text{étale}} & & \downarrow \\ Y & \hookleftarrow & Y_0 \end{array}$$

defined by sheaf of nil ideals.

$$X \longrightarrow X_0$$

$$\text{Étale}/Y \longrightarrow \text{étale}/Y_0$$

is equivalence of categories.

~~March 26~~ Feb 26 No lecture March 3.

Add to Def 3

To give description of ~~f locally finite~~ presentation.

$$Z_i = \operatorname{Spec} A_i \longleftarrow Z_j \qquad \text{affine (affine lying over } Y) \qquad Z = \operatorname{Spec} A$$

$$A_i \longrightarrow A_j \qquad\qquad A = \varinjlim A_i$$

$$X(A_i) \longrightarrow X(A')$$

So

$$\varinjlim X(A_i) \longrightarrow X(A)$$

f locally of finite presentation $\Leftrightarrow$ for all such choices,) have iso here

Covering of Y.

$$\begin{array}{c} X \\ \downarrow f \\ Y \end{array}$$

f is finite ie $X \cong \mathrm{Spec}(\mathcal{A})$

$\mathcal{A}$ quasi-coherent algebra, such that $\mathcal{A}$ is finite over $\mathcal{O}_X$

ie ~~finitely generated as~~ ~~sheaf of~~ module of finite type.

étale covering = étale + covering.

$\Updownarrow$ evident

f finite & locally free, $\quad X_{\bar{y}} \cong \coprod_{i\ \text{finite}} \mathrm{Spec}\, k(\bar{y})$

Spec $\mathcal{A}$ as above, but with $\mathcal{A}$ locally free of finite type.

as before

$\Updownarrow$

$\mathcal{A}(y) = \mathcal{A}_y \otimes k(y) = \prod_i k_i$

k_i finite separable extensions of $k(y)$.

$\Updownarrow$

~~$\exists$~~ $\forall y \in Y$, $\exists\, \mathcal{U} \ni y$, $\mathcal{U}' \xrightarrow{\text{f.f.}} \mathcal{U}$ ($\mathcal{U}'$ affine)

such that $X' = X \times_Y \mathcal{U}'$ is a constant scheme with finite fibers. (constant $X' \cong I_{U}$, I finite set.)

$\Updownarrow$

> EGA IV [illegible] of faithfully flat base change discussed at length

$\exists\ Y' \longrightarrow Y$ such that
étale
finite
surjective

$$\begin{tikzcd} Y'\times_Y X = X' \arrow[r] \arrow[d] & X \arrow[d] \\ Y' \arrow[r] & Y \end{tikzcd}$$

$Y' = \coprod_{n\geq 0} Y'_n$

and $X'|Y'_n \cong I_n\, Y'_n$

(n: rank of locally free sheaf at A)

to find Y': Assume $n\geq 1$ and take $Y' = X$ as 1^{st} step induction.
is not good enuf.

$$\begin{tikzcd} X \arrow[d, "f"'] & X' \arrow[l] \arrow[d] \\ Y & X = Y' \arrow[l] \arrow[u, bend right, "\text{diagonal} : \delta"'] \end{tikzcd}$$

Y'' ($n-1$ copies of Y')
$\downarrow$
Y''

f étale $\Rightarrow$ diagonal is ~~open immersion~~

also is closed immersion.

$\therefore\ X' = \underset{\delta(Y')}{\overset{\shortparallel}{\Delta}} \coprod X'_1$

X'_1 étale covering of rank $n-1$.

By induction assumption, can make X'_1 constant by base change.

$\boxed{\underline{\mathrm{Isom}}(\underline{I}_{nY}, X) \text{ defined by } Z \longmapsto \mathrm{Isom}(\underline{I}_{nZ'}, X_Z) \text{ represented by } Y''.}$

(5) $\eta \in Y$ normal irreducible

$\uparrow$ ~~finite~~, every irreducible component of X dominates Y.

X

$y \longmapsto \mathrm{Card}\, X(\bar{y})$

étale means this function is locally constant. (continuous)

and X_η is étale over η. (finite product of separable extensions).

η = generic point

Standard Example $\quad E'_K \longrightarrow E'_K$

$x \longrightarrow x^n$

K any ~~basis~~ alg. closed field (n invertible) $i \nmid n$, char K.

For complement of origin, is étale covering. "Ramified" at origin.

Ramification not understood, even char 0.
(Good theory of ramification group commutative Serre)

$\mathcal{C} = \mathrm{Cov\acute{e}t}(Y)$ = category of étale coverings.

if $Y = \coprod_i Y_i$

then $\mathrm{Cov\acute{e}t}(Y) \cong \prod \mathrm{Cov\acute{e}t}(Y_i)$

so assume Y connected.

$$\begin{array}{c} Y \\ \uparrow \\ \xi \end{array} \qquad \mathcal{C} \longrightarrow \mathrm{Cov\acute{e}t}(\xi) \quad \text{inverse image}$$

$\xi \to Y$

$\xi = \mathrm{Spec}(\Omega)$

Ω alg. closed.

equiv.

$\mathcal{C} \longrightarrow$ finite sets.

$X \xrightarrow{F_\xi} X(\xi)$

Fibre functor at ξ.

$$\mathrm{Aut}(F_\xi) = \pi_1(Y, \xi)$$

$= \pi$ has topology making it profinite group.

For all coverings X,

$$\pi_1(Y, \xi) \xrightarrow{\text{injective}} \prod_X \overbrace{\mathrm{Aut}\, F_\xi(X)}^{\text{finite group}}$$

give topology induced by this product (direct topology on each group.)

Its closed subgroup of the product. Hence is profinite group.

Functor. $\mathcal{C} \xrightarrow{F_\xi} (\Pi\text{-finite sets})$

finite sets on which π acts continuously.

Then Thm 1. F_ξ is equivalence of categories

Thm 2. If ξ, ξ' are two geometric pts (at necess. same characteristic)

Then $F_\xi \simeq F_{\xi'}$ (not necess. unique isomorphism).

Reference SGA 1 V.

To get group scheme étale covering of Y, is just ~~group~~ finite group on which Π acts: etc

March 5

$\boxed{\text{Have defined}}$

étale schemes.

étale coverings

$$\begin{array}{c} X' \\ \downarrow \\ X \end{array}$$

X' connected

$$\begin{array}{c} X' \\ \downarrow \\ X \end{array}$$

Spec k $= x \longrightarrow X$

k alg. closed.

geometric pt.

$$\begin{array}{ccc} \text{cov ét} & (\hat{a}) \to & \text{finite sets} \\ (\text{étale coverings}) & \xrightarrow{\text{geometric pts?}} & \text{finite sets} \end{array}$$

forget-Sol.

$$X' \xrightarrow{\varphi_x} X'(x)$$

Fu. $\pi_1(X, x) = \mathrm{Aut}(\varphi_x)$ profinite group.

Thm (a) is equivalence of ~~…~~ categories.

finite and infinite inverse systems co-incide in the two categories. etc.

$$X' \times_X X' \longrightarrow X' \quad \text{(operation on coverings)}$$

iso. classes of connected coverings $\longleftrightarrow$ finite sets on which π acts transitively

Also have $\varphi_x \simeq \varphi_{x'}$ (even if characteristics are different)

so $\pi_1(X, x) \cong \pi_1(X, x')$ $\boxed{\text{"Galois theory"}}$

Review $\begin{array}{c} Z \\ \downarrow \\ X \end{array}$ étale and finite (étale covering)

if you can cover X by $U_i = \operatorname{Spec} A_i$, and for each A_i $\exists$ faithfully flat morphism $X_i \longleftarrow X_i'$ (affine ~~too~~)

finite set.

$$
\begin{tikzcd}
Z \arrow[d] & Z_i \arrow[d, dashed] & & Z_i' \cong (E_i)_{U_i'} \arrow[d] \\
X \supset & U_i & & U_i' \arrow[ll, "\text{f.f.}"']
\end{tikzcd}
$$

(can take U_i equal to X if X affine.)

equiv flat, finite, fibres separable. (X locally noetherian) finite presentation etc.

$\boxed{\text{over } \mathbb{C}, \text{ étale covering } = \text{ topological covering space.}}$

[Is usual galois theory if $X = \operatorname{Spec} k$.

Same as theory with fundamental group of topo. ("prodiscreet" groups)

→ Take an alg pt. the alg closure Ω of k.

(finite product) $\prod K_i$ = ét cov.

K_i: finite separable extension of k.

$k_i \longrightarrow \Omega$

In classical galois theory we studies intermediate extensions of a single K_i.

$$A = \prod K_i$$

$$K \longrightarrow \Omega$$

π - ~~…~~ $\subset \pi_i$

To K' ~~…~~ $\longrightarrow \mathrm{Hom}_{K\text{-alg}}(K', \Omega) = E$ — π_0 acts on E translation

$$E \to F \to 1$$

K' corresponds to F "quotient set"

$E \Rightarrow \pi_0$ finite group. acts on E.

Seek all quotient sets of E, (on which π operates)

~~such are~~ quotients of π by normal subgroups.

$$X' \to X, \quad X'^{an} \to X^{an}$$

étale here
local homeomorphism.
finite covering space.

Functor $\mathrm{Et}(X) \longrightarrow \mathrm{Covet}(X^{an})$

only finite coverings

Riemann's existence thm says this is equivalence of categories. [Riemann dim 1
Grauert - Remmert higher dim'l $\boxed{\text{no one understands proof}}$
Serre (GAGA) if X compact (10 years ago)

but simpler proof now using resolution of singularities SGA 1 XIV in Springer Lecture notes. (forthcoming).
Mme Raynaud.

X alg. scheme over $\mathbb{C}$.

$\underline{S}$ analytic space

$\mathrm{Hom}(\underline{S}, X)$

admissible ~~hom of~~ ~~[illegible]~~ locally ringed $\mathbb{C}$-algebras.

(homs on stalks are local homomorphisms)

This functor is representable — gives analytic structure on X, X^{an}.

$$\begin{tikzcd} X & X^{an} \arrow[l, "\text{universal}"'] \\ & \underline{S} \arrow[u] \arrow[ul] \end{tikzcd}$$

Show representable for open sets. (patch together).

$X = \bigcup X_i \qquad X_i^{an}$ exists $\to X^{an}$ exists.

If $Y \subset X$ closed subscheme, if X^{an} exists then Y^{an} exists (trivial).

for $\left(\begin{array}{cc} Y & X^{an} \\ \mathcal{I} & \mathcal{I}\mathcal{O}_X \\ \text{coherent sheaf of ideals over } X & \text{defines } Y^{an} \end{array} \right)$.

Now we reduced to E^n

but if $X = \prod X_i$

X^{an} exists if X_i^{an} ~~are~~ exist.

Now we reduced to E^1. $(E^1)^{an}$

$\mathrm{Spec}\, \mathbb{C}[t] = E^1 \leftarrow X = S$

$\updownarrow$

~~~~sections~~ Section of structure sheaf

$\updownarrow$ representable

eg analytic space $\mathbb{C}$. $(E^1_{\mathbb{C}})^{an}$.
~~~~

covering but not geometric notion.

$$\begin{array}{c} X' \\ \downarrow \\ X \end{array}$$

covering = finite morphism

strict covering = for all irreducible components X_i' of X', X_i' dominates an irreducible component X_j of X.

locally of finite type over $\mathbb{C}$

X' normal

X normal.

$$\begin{tikzcd}
X^{an} \arrow[d, "\text{strict finite}"'] & X'_{\text{normal}} \arrow[l] \arrow[d, "\text{strict finite}"] \\
X^{an}_{\text{normal}} \arrow[r] & X_{\text{normal}} \arrow[l]
\end{tikzcd}$$

Thm $X' \longrightarrow X'_{an}$

R.E.T. strict normal coverings of X $\xrightarrow{\alpha}$ strict normal finite coverings of X^{an}

is fully faithful ; if X complete (X^{an} compact) then α is an equivalence if X not complete — essential image

= those coverings X' of X^{an}

such that ramification ~~locus~~ set $\underline{\underline{D}}$

is algebraic (eg if unramified).

{ we described by algebraic eqns } ~~not~~ ~~not~~ (not a pt in essential image)

eg can take 2 sheeted covering of $\mathbb{C}$

ramified at pts of $\mathbb{Z}$. . . . .

Not algebraic ~~branch~~ ramification set.

(goes wrong at infinity!)

ie $\forall$ $\underline{D}$ $\exists$ $D \subset X$ and

that $\underline{D} = \varphi_X^{-1}(D)$

if X complete — is main thm of GAGA

F — $\to$ $F^{an} = \varphi_X^*(F)$ $\quad$ F coherent $\Rightarrow$ F^{an} is coherent.

$X \xleftarrow{\varphi_X} X^{an}$

[not fully faithful, more analytic sections if X not complete.]

Serre says $\overset{X \text{ complete}}{\Rightarrow}$ is equivalence of categories.

$$\mathrm{Coh}(X) \approx \mathrm{Coh}(X^{an})$$

$F \qquad \mathcal{F}$ (compatible with $\otimes$, $\oplus$)

$$\begin{array}{ccc} F \otimes F & & \mathcal{F} \otimes \mathcal{F} \\ \downarrow & \text{same as} & \downarrow \\ F & & \mathcal{F}' \end{array}$$

Coverings in either case correspond to coherent sheaves with algebra structure (Spec) [presumably]
∴ get them in compact case.

General Case.

If $\mathcal{U}$ dense in X (normal), Riemann's existence then true in $\mathcal{U}$.

$\mathcal{X}'$ Then true in X.

$$\begin{array}{ccccc} X' & & \mathcal{U}' & & \mathcal{X}' \\ \text{seek} & & \downarrow & & \downarrow \text{normal struct} \\ X & & \mathcal{U}^{an} & \subset & X^{an} \end{array}$$

$\mathcal{U}'^{an} = \mathcal{U}'$. Use normalization of X in function field of $\mathcal{U}'$. gives X'. X' gives rise to X'^{an}.

~~And~~ Restricted to $\mathcal{U}^{an}$ is iso to $\mathcal{X}'$.

is general fact about analytic spaces that must $\bar{X}' = X'^{an}$.

X' non singular

$\therefore$ can assume X to be affine and non-singular.

proper birational $\pi \downarrow$ restricted to $\mathcal{U}$. $\pi^{-1}(X - \mathcal{U})$ has normal crossings)

$\bar{X} \supset \mathcal{U} \subset \mathbb{P}^r_{\mathbb{C}}$ $\qquad$ $\bar{X}$ complete.

affine non singular

Hironaka. $\exists X'$

$X = \bar{\mathcal{U}}$ $\qquad$ $X' - \mathcal{U} = D$ is divisor with normal crossings.

(~~complete~~ described analytically by $x_1 x_2 \cdots x_r = 0$)

eg

if $r = 2$.

$\mathcal{U} \subset X$

$\mathcal{U}' \qquad \bar{X}'$

$\downarrow \qquad \downarrow$

$\subset X$ ~~non singular~~

get can extend.

analyse locally

Show extension is unique, can do locally then in analytic spaces

calculate fundamental group of complement and use dictionary and reduces to extending certain cases.

e.g. Z^r compact

$\mathcal{U}' \subset X' \leftarrow ?$

seek $\downarrow \qquad \downarrow$

$\mathcal{U} \subset X'$ non-singular compactification from above. (complement of $\mathcal{U}$ nice singularities)

what comes from $\bar{X}$

By the for complete varieties get $\bar{X}'$ restrict to $\mathcal{U}'_{\bar{X}}$.

March 10

$$D^n \supset D^{*r} \times D^{n-r} = \mathcal{U}$$

[in analytic case ex usual fundamental group]

$$\pi_1(\mathcal{U}) = \mathbb{Z}^r$$

$$\left(\mathbb{Z}/N\right)^r \qquad E = \Gamma/\Gamma' \qquad E^{1}/\Gamma'$$

$$\begin{array}{ccc}
\mathcal{U}_N & & D^r \times D^{n-r} = D^n \\
\Big\downarrow {\scriptstyle \Gamma = (\mathbb{Z}/n\mathbb{Z})^r} & & \Big\downarrow \\
D^{*r} \times D^{n-r} = \mathcal{U} & \hookrightarrow & X = D^r \times D^{n-r} = D^n
\end{array}$$

reduced to extending $\mathcal{U}_N$

analogous to ... given explicitly.

$$\begin{array}{c}
X' \\
\Big\downarrow {\scriptstyle \text{R.E.T.}} \\
X \\
\Big\downarrow \\
C
\end{array}
\qquad X^{an}$$

small

locally finite type

(finitely sheeted)

$$\mathrm{Cov\acute{e}t}(X) \xrightarrow{\simeq} \mathrm{Cov\acute{e}t}(X^{an})$$

Relation between algebraic and ~~al~~ usual fundamental groups.

category of ~~[illegible]~~ X' classified by $\pi_1(X, x)$ acting on finite sets

$\mathrm{Cov\acute{e}t}(X^{an})$ [not finite] represented by $\pi_1(X^{an}, x)$ acting on sets (not finite)

$\mathrm{Cov\acute{e}t}_{fin}(X^{an}) = \pi_1(X^{an}, x)$ [usual fundamental group] acting on finite sets.

(continuous representations of $\hat{\pi}_1$) — profinite ~~completion~~ compactification

$\pi_1(X^{an}, x) \cong \hat{\pi}_1(X^{an}, x).$

$$\begin{tikzcd} X' \arrow[d] \arrow[r] & Y' \arrow[d] \\ X \arrow[r] & Y \end{tikzcd}$$

Y' étale

X' inverse image

geometric pt of X

geo pt of Y.

$\exists$ homo. of profinite groups $\pi_1(X, \xi) \longrightarrow \pi_1(Y, \xi)$ so that is induced by restriction of scalars. (general nonsense)

$Et(X) \xleftarrow{f^*} Et(Y)$

$\wr\wr$

finite $\pi_1(X, \xi)$ sets (continuous action).

finite sets on which $\pi_1(Y, \xi)$ acts continuously.

$$\mathrm{Aut}(\varphi_x) = \pi_1(X, x) \longrightarrow \pi_1(Y, y) = \mathrm{Aut}(\varphi_Y).$$

~~Fibre functor of~~ $\varphi_Y = \varphi_x \circ f^*$

$\therefore$ every auto of φ_x defines

automorphism of φ_Y.

check corresponds to f^* under dictionary.

$\pi_1(X, x) \longrightarrow \pi_1(Y, y)$ epi

$\Leftrightarrow$ f^* fully faithful etc.

More generally, S any connected, simply connected scheme ($\neq \emptyset$)

(all étale coverings trivial)

Then can ~~do~~ do theory with $S \xrightarrow{x} X$

$\pi_1(X, x)$

gives functor $\mathrm{Cov\acute{e}t} \longrightarrow \mathrm{Cov\acute{e}t}(S)$

$\|$

finite sets.

Simply Connected $\mathbb{P}^n_k$, complete-intersection

k alg. closed. any characteristic

affine space. char 0.

not char p.

Covering $X \longrightarrow X^p - X$

covering of G_a with galois group $\mathbb{Z}/p\mathbb{Z}$.

$$\pi_1(G_a, 0) \xrightarrow{\text{epi}} \mathbb{Z}/p\mathbb{Z}.$$

in fact huge fundamental group

(commutative part covered by geometric class field theory — .)

(Guess: In char p. no affine variety can be simply connected)

Prove $\mathbb{P}^1$ simply-connected (any char) (field alg. closed — otherwise get fundamental group of K).

X' étale

$$\downarrow$$

$X = \mathbb{P}^1$.

$$g = \dim H^1(X, \mathcal{O}_X)$$

$$\chi(X) = 1-g$$

χ multiplicative

Hurwitz formula for étale coverings

$$\chi(X') = d\,\chi(X) = d$$

$$\chi(X') = 1-g'.$$

d at least one, so g' must be zero also.

$$\begin{tikzcd} x \arrow[dr] & & \\ & X \arrow[d, "\text{rational pt gives splitting.}"'] & \bar{X} \arrow[l] \\ & \operatorname{Spec} K & \bar{K} \arrow[l] \end{tikzcd}$$

$$\begin{tikzcd} 1 \arrow[r] & \pi_1(\bar{X}, x) \arrow[r] & \pi_1(X, x) \arrow[d] \\ & & \pi_1(K, x) \arrow[d] \\ & & 1 \end{tikzcd}$$

$\bar{K}$ = alg. closure of K.

exact sequence.

Split if $\exists$ rational point of X/K.

but action not trivial on $\pi_1(\bar{X}, x)$. semi-direct product.

If $\pi_1(\bar{X}, x)$ trivial, get ~~$\pi_1(\bar{X},$~~ $\pi_1(K, x) = \pi_1(X, x)$.

eg for $\mathbb{P}^1_k$ k not alg. closed.

If fundamental group abelianized — tied up with Riemann hypothesis etc deep.

Situation worse.

~~Sch~~ Schemes.

$S \xleftarrow{f} S'$ quasi-compact

if inverse image of quasi-compact set is quasi-compact.

$V(U_i)$

~~$V(U_i)$~~ U_i quasi-compact $\qquad f^{-1}(U_i)$ — quasi-compact.

S q.c. and f — q.c. $\Rightarrow$ S' quasi compact.

S quasi-compact $\Leftrightarrow$ finite union of affines.

Eg

Not quasi-compact (a) $\quad U^I \longrightarrow U \quad$ I infinite.

(b) $\quad X \longleftarrow \coprod_{x \in X} \operatorname{Spec} k(x)$.

eg X affine line.

(not q.c. except if X finite)

Assume quasi-compact and faithfully flat.

$S \longleftarrow S'$

Descent statement (if true after f.c. f.f. base extension then true before).

proofs EGA IV §2.

F quasi-coherent on S

F' finite type $\longrightarrow$ F finite type

F' finite presentation $\longrightarrow$ F finite presentation.

F' locally free finite type $\Longrightarrow$ same for F (with same rank)

$\longrightarrow$

Immediate to reduce to S, S' affine (all are local properties).

ie/ F' flat $\longrightarrow$ F flat

F' faithfully flat $\longrightarrow$ F flat

$S \longleftarrow S'$

affine.

$\parallel$

$\bigcup S'_i$

affine

$S'' = \coprod S'_i$

union of affines.

S'' faithfully flat over S.

disjoint union of affines

$$S \longleftarrow S'$$

Now $A \longrightarrow A'$ $\quad A'$ f.f. algebra over A.

geometric statements reduced to algebraic.

e.g. $M \longrightarrow M' = M \otimes_A A'$ $\Big\}$ some statements true which in non-commutative case ("sheaf of algebras")

M' finite type over A' $\Rightarrow$ M finite type ~~over~~ A etc. (Bourbaki)

~~X~~

Schemes over S. $\qquad$ over S'.

(cartesian diagram)

$$\begin{tikzcd} X \arrow[d, "f"] & X' \arrow[l] \arrow[d, "f'"] \\ Y & Y' \arrow[l] \\ S & S' \arrow[l] \end{tikzcd}$$

change notation to get ~~...~~ $Y = S$

(or forget)

$S \longleftarrow S'$ qc and f.f.

~~...~~ in $Y \longleftarrow Y'$

f' quasi compact $\Rightarrow$ f q.c.

f' finite type $\Rightarrow$ f finite type $\Big\}$ true if noetherian

[f' finite presentation $\Rightarrow$ f finite presentation]

f' immersion $\Rightarrow$ f immersion

also closed $\qquad$ also closed.

(iso - surj. for immersion).

f' surjective $\Rightarrow$ f surjective

f' finite $\Rightarrow$ f finite.

f' proper $\Rightarrow$ f proper. (finite type, separated and closed)

f' flat $\Rightarrow$ f flat.

f' f.f. $\rightarrow$ f f.f.

f' étale $\Rightarrow$ f étale. (from above).

separable $\Rightarrow$ separable.

unramified $\Rightarrow$ unramified

smooth $\Rightarrow$ smooth

(usual properties stable under base change descend).

$\boxed{\text{hence deep.}}$

(can assume S, S' affine and $Y \equiv S$.)

Also can make statements about X' itself.

X' regular $\Rightarrow$ X regular but not conversely

If have f.f. f.c. $X \leftarrow X'$ then X has quotient topology.

Z fin $\Leftrightarrow$ $f^{-1}(Z)$ open.

closed $\Leftrightarrow$ closed.

$f^{-1}(\overline{Z}) = \overline{f^{-1}(Z)}$ (need only f flat here). used
local on X'

Base change is type of localization

(~~above are local for g.c. f.f. topology~~)

Times | Thursday 2:30 – 4 | office 4–6.
wednesday cancelled.

Friday 10:30–12:30.

March 11 Public Lecture Grothendieck

Relations between algebraic geometry and arithmetic (topology ...).

X compact, locally contractible top. space finite dimension

$H^*(X)$ cohomology ring coefficients $\mathbb{Q}$. $H^0(X) = \mathbb{Q}$. X connected

Properties ① anti-commutative graded algebra over $\mathbb{Q}$. all co-incide. { - singular - Čech - simplicial

~~Proper~~ ② $H^i(X) = 0$ unless $0 \le i \le \dim X$

③ $X \longmapsto H^*(X)$ is contravariant functor wrt X.

$X \xrightarrow{f} Y$ induce $H^*(X) \xleftarrow{f^*} H^*(Y)$

④ $\dim H^*(X) < \infty$

$H^i(X)$ degree i. (can talk about trace, det etc. in each i induced by f^* $f: X \to X$. etc.

③
$$\begin{tikzcd} & X \\ X \times Y \arrow[ur] \arrow[dr] & \\ & Y \end{tikzcd}$$

$$H^*(X) \otimes H^*(Y) \xrightarrow{\approx} H^*(X \times Y)$$

isomorphism
Kunneth

④ $X = \coprod_\alpha X_\alpha$

$$H^*(X) \xrightarrow{\approx} \prod H^*(X_\alpha)$$

⑤ Assume X is manifold (orientable) dim n. connected.

$$\eta_X \in H^n(X) \cong Q$$

η_X canonical
X as cycle ~~one~~ over itself.

$$H^i(X) \times H^{n-i}(X) \to H^n(X) \cong Q$$

perfect pairing
(i.e. each of the two spaces is dual of the other)

(for varieties, ~~have Gysin homo~~

$$H^i(Y) \xrightarrow{f^*} H^i(X)$$

$$H^{n-i}(X) \xrightarrow{f_* = (f^*)^*} H^{m-i}(X)$$

$$H^{n-i}(X) \to H^{m-n+i}(Y)$$

(6) X manifold.

cup product formula.

f_* Gysin homomorphism (dual of f^* using Poincaré duality thm).

$$H^*(X) \xrightarrow{f_*} H^*(Y)$$

f_* not ring hom.
just of vector spaces over $\mathbb{Q}$.

increases degrees! by $m-n$.

$\dim X = n$
$\dim Y = m$

$$H^i(X) \xrightarrow{f_*} H^{i+m-n}(Y)$$

$$1_X \longmapsto c(f) \in H^{m-n}(Y).$$

$1_X \in H^0(X)$

$c(f)$: fundamental class of f.

called also $c(X)$ if f is inclusion

$X, X' \subset Y$

codim d, d'.

$X \cap X'$ codim $d+d'$
(assume intersection general enough)
"transversal"

$c(X) \in H^d(Y)$ $\quad$ $c(Y) \in H^{d'}(Y)$

Then $c(X \cap X') = c(X)\,c(X')$ $\in H^{d+d'}(Y)$
product in H^*.

If intersection not transversal — can give meaning perhaps "intersection multiplicity" more intricate.
so long as codimension has correct intersection.

7) Lefschetz Formula

$f : X \to X$ endomorphism, X compact manifold dim n

$$\nu(f) = \sum (-1)^i \operatorname{Tr} f^*_{H^i(X)}$$

no. of fixed pts. (counted with multiplicity)

assume also fixed pts isolated.

$$\begin{array}{ccc} X \times X & \hookleftarrow & \Delta_X \\ \cup & & \\ \Gamma_f & — & \Gamma_f \cap \Delta_X \end{array}$$

fixed pts : intersection

multiplicities attached to intersection pts as indicated above

easy from above formal properties.

∴ (7) is particular case of (6).

$$= Z_i(t)$$

$$\det\left(1 - f^*_{H^i(X)}\, t\right) = \prod_j \left(1 - \alpha_{ij} t\right)$$

$$\nu(f) = \sum_{i,j} (-1)^i \alpha_{ij}$$

α_{ij} are algebraic integers.

$$c_d = \nu(f^d) = \sum_{i,j} (-1)^i \alpha_{ij}^d$$

$\sim f^d$ "periodic pts" of order d

def $$Z_f(t) = \sum_{d \geq 1} c_d\, t^{d-1} = \sum_{d \geq 1} \nu(f^d)\, t^{d-1}$$

$$= \sum_{d,i,j} (-1)^i \alpha_{ij}^d\, t^{d-1}$$

$$= \sum_{i,j} (-1)^i \left(\sum_{d \geq 1} \alpha_{ij}^d t^{d-1} \right) = \sum_{i,j} \frac{\alpha_{ij}}{1 - \alpha_{ij} t}$$

$$\frac{\alpha_{ij}}{1-\alpha_{ij}t} = D \operatorname{Log}\left(\frac{1}{1 - \alpha_{ij} t}\right)$$

$$Z_f(t) = \frac{L_f'(t)}{L_f(t)}$$

where $L_f(t) = \prod_{ij} \left(\frac{1}{1-\alpha_{ij} t} \right)^{(-1)^i}$

vary j

$$= \prod_i \left(\frac{1}{\prod_j (1-\alpha_{ij} t)} \right)^{(-1)^i}$$

$$= \prod_i Z_i(t)^{(-1)^{i+1}}$$

$$= \frac{Z_1(t)\ Z_3(t) \cdots}{Z_0(t)\ Z_2(t)\ Z_4(t) \cdots}$$

need there not be a $Z_0(t)$, $1-t$.

$\dim X = n$.

$H^n(X) \cong \mathbb{Q}$.

Poincaré duality

$f^*_{H^n(X)} = d(f)$

degree of f

(integer).

"

no. of times

X covers itself.

$$d\,\Big(\underbrace{{}^t f^*_{H^i(X)}}_{}\Big)^{-1} = f^*_{H^{n-i}(X)}$$

$\longleftarrow$ dual spaces $\longrightarrow$

if $d \neq 0$

~~f^* known if known half the time~~.

$\Downarrow$

$\alpha_{i1} \; \cdots \; \alpha_{ib_i}$ eivs for f^* on $H^i(X)$

$\alpha_{n-i\,1} \; \cdots \; \alpha_{n-i\,b_{n-i}}$ f^* on $H^{n-i}(X)$.

can associate pairwise, so product $\alpha_{ij}\,\alpha_{n-i\,j} = d$.

gives $Z_f\left(\frac{1}{t}\right) = \overbrace{(\ )}^{\text{some factor.}} \cdot Z_f(t)$

ξ = hyperplane section

$f^*(\xi) = c(\xi)$

Assume X complex manifold $\hookrightarrow P^r_{\mathbb{C}}$.

(complex dim n)

Real dim $2n$.

$f: X \to X$ endomorphism. for complex structure

$H^i(X)$ $\quad i \leq n$.

$(\alpha_{ij}) \longrightarrow \left(\frac{c^i}{\alpha_{ij}}\right)$ two sequences up to order are the same.

(Hodge theory gives this result).

Weil apply above to arithmetic

X $\quad$ card $\mathbb{Q}(X(\mathbb{F}_q))$ (no of solutions of diophantine eqns)

$\downarrow$ $\quad$ card $(X\,\mathbb{F}_{q^d}) = \mathcal{C}_d(X)$.

$\mathbb{F}_{q^d} \longrightarrow \mathbb{F}_q$ (field with q elts).

Interpretation study of X : study of the c_d.

<u>Weil conjecture</u> $$Z_X(t) = \sum c_d(X)\, t^{d-1}.$$

is log derivative of rational function ~~also~~ has properties of series arising in topology also.

<u>Riemann Hypothesis</u> $$|\alpha_{ij}| = q^{\frac{i}{2}}$$

<u>motivation</u>

$$\begin{tikzcd} X \arrow[r, dash] \arrow[d, dash] & \bar{X} \arrow[d, dash] \\ K = \mathbb{F}_q \arrow[r, dash] & \bar{K} \end{tikzcd}$$

$\mathbb{F}_{q,X}$ frobenius endo. of $\bar{X}$.

$X(\mathbb{F}_q) =$ fixed points of $\mathbb{F}_{q,X}$

$$\mathbb{F}_q : (x_1, \ldots, x_N) \longrightarrow (x_1^q, \ldots, x_N^q)$$ in affine space

$$\text{fixed pts of } \mathbb{F}_{q,X}^d = X(\mathbb{F}_{q^d})$$

~~Try to find cohomology~~ for Étale cohomology theory (Artin, Grothendieck).

cohomology theory ~~for~~ for varieties over F alg closed char p.

gives Weil conjectures.

but not Riemann hypothesis (known in some cases

any curves, | not surfaces

Weil - abelian varieties

For curve X / $\mathbb{F}$ proj non singular curve.
irreducible over $\mathbb{F}$.
rank $H^1 = 2g$.

$$\underbrace{\operatorname{card} X(\mathbb{F}_q)}_{\nu} = 1 - \sum \alpha_{ij} + q$$

use Riemann hypothesis which is proved here.

deep.
$$\boxed{|\nu - 1 - q| \leq 2g\sqrt{q}}$$

no of pts of proj line.

constant.

smaller factor than $q+1$.

very sharp estimate.

(can't get by purely analytic methods!)

Bibliography (1) Lang's Book. on abelian varieties

(2) Weil "Sur les courbes algébriques".
hard because before algebraic geometry.

(3) Grothendieck - Castelnuovo's inequality on Surfaces. easier than (2).

related to Hodge index theorem.

Ring variety ⇒ Artinian (Yanagihara)

March 12 $\quad C$ = category (eg open subsets of topological space)

Topology on C ?

Assume C admits fibred products

$$\begin{tikzcd} X \arrow[r, leftarrow] \arrow[d, dashed] & X' \arrow[d, dashed] \\ S & S' \arrow[l, hook'] \end{tikzcd}$$

ie X' exists in C

Covering family $\quad \left\{ S_i \xrightarrow{f_i} S \right.$

Axioms
(0) identity is covering.
(1) Transitivity
(2) Stability by base change.

$$\begin{tikzcd} S_i \arrow[d, "f_i"] & S_i' \arrow[l, dashed] \arrow[d, "f_i'"] \\ S & S' \arrow[l] \end{tikzcd}$$

f_i covering $\Rightarrow f_i'$ covering

eg C = open subsets of top. space

(1) Transitivity

$$\begin{tikzcd} & S_{ij} \arrow[d, "g_{ij}", "\text{covering}"'] \\ S_i \arrow[d, "f_i"', "\text{covering}"] & S_i \\ S & \end{tikzcd}$$

Then $f_i \circ g_j$ cover S.

(3)
$$\begin{tikzcd} S_{i\alpha} \arrow[d] & \\ T_\alpha \arrow[r] & S \end{tikzcd}$$

If cover locally, then cover.

I.e., $T_\alpha \longrightarrow S$ cover. Then if $\forall \alpha$
$$\begin{tikzcd} S_{i\alpha} \arrow[d] \\ T_\alpha \end{tikzcd}$$
cover T_α, then
$$\begin{tikzcd} S_i \arrow[d] \\ S \end{tikzcd}$$
cover S.

C + Topology is called <u>Site</u>.

$$\hat{C} \supset \tilde{C}$$

"presheaves" — category of sheaves.

F presheaf (functor $C^{\circ} \longrightarrow \mathrm{Sets}$).

(a) F separated if for every covering family
$$\begin{tikzcd} S_i \arrow[d] \\ S \end{tikzcd}$$

$$F(S) \longrightarrow \prod_i F(S_i) \quad \text{is injective.}$$

(b)
$$\begin{tikzcd}[column sep=small] & S_i \times S_j = S_{ij} \arrow[dl] \arrow[dr] & \\ S_i \arrow[dr] & & S_j \arrow[dl] \\ & S & \end{tikzcd}$$

sequence is ~~exact~~ exact

$$\cancel{F(S)} \quad F(S) \longrightarrow \prod_i F(S_i) \longrightarrow \prod_{i,j} F(S_{ij})$$

(S_i)

Properties of Sheaves

① $\varinjlim$ and $\varprojlim$ exist.

(trivial)

↓

take inverse limit argument wise

$$\left(\varprojlim F_i\right)(S) = \varprojlim F_i(S)$$

$\varprojlim F_i$ is sheaf again, and is inverse limit in category of sheaves.

Take $\varinjlim_i F_i$ of presheaves. $\underbrace{\left(\varinjlim_i F_i\right)}_{P}(S) = \varinjlim_i F_i(S)$

is just a presheaf. P.

$\varinjlim_{\tilde{C}} F_i$

$$\begin{tikzcd} P \arrow[rr] \arrow[dr] & & F \\ & \tilde{P} \arrow[ur] & \end{tikzcd}$$

seek $\tilde{P}$, sheaf
universal wrt
map of presheaf P into sheaf F

Presheaf can be associated to pre sheaf
here also.

Take $\varinjlim_{\tilde{C}} F_i = \tilde{P}$.

2)

$$\begin{tikzcd} & F_i \arrow[d, "u_i"] \\ \coprod F_i \arrow[r] & F \end{tikzcd}$$

$\coprod$ = direct sum. categor'l sense (defined by $\boxed{1)}$)

in sets = disjoint union

sums are
disjoint.

ϕ: initial - object
(in sets = empty set)

$$F_i \times_F F_j = \begin{cases} \phi_{\tilde{C}} & i \neq j \\ F_i & i = j. \end{cases}$$

ie u_i is a
monomorphism.
$\Updownarrow$
diagonal is
monomorphism.

(b) Sums are universal

$$\begin{tikzcd} F_i \arrow[d, "f_i"'] & F_i' \arrow[l] \arrow[d, "f_i'"] \\ F & F' \arrow[l] \end{tikzcd}$$

If $F = \coprod F_i$ by f_i

then $F' = \coprod F_i'$ by f_i'.

(clear for sets).

(base change commutes with inverse limits in general — above is commuting with certain direct sum).

<u>Complement</u>

$$\tilde{C}/S \longrightarrow \tilde{C}/S'$$

$$X \longmapsto X \times_S S'$$

$$S \longleftarrow S'$$

commutes with arbitrary direct limits as well as inverse.

(3) Equivalence relation.

$$R \hookrightarrow F \times F$$

$\forall S$, $R(S) \subset F(S) \times F(S)$ is equivalence relation

$$\begin{tikzcd} R \arrow[r, shift left, "a"] \arrow[r, shift right, "b"'] & F \arrow[r] \arrow[dr, "f"'] & Q \arrow[d, dashed] \\ & & G \end{tikzcd}$$

Q cokernel

If $f \circ a = f \circ b$ then factor uniquely thru Q.

Q exists since (1)

Then $Q = F/R$.

Have: $R \longrightarrow F \times_Q F \subset F \times F$

in fact $R \cong F \times_Q F$ (R is effective equivalence relation).

4) $\exists$ a set of generators $(F_i)_{i \in I}$ of $\tilde{C}$.

~~[illegible]~~ ie

$$\begin{tikzcd} & F_i \arrow[dl] \arrow[dr] & \\ G \arrow[rr, hook, "u"] & & H \end{tikzcd}$$ in $\tilde{C}$.

if every morphism $F_i \longrightarrow H$ (same i)

factors through G, then $G \xrightarrow{\sim} H$.

or if u not iso, then $\exists\, i$,

$F_i \longrightarrow H$ not factoring through G.

Then F_i generating family.

Claim $\tilde{C}$ has generating family.

$S \in$ obj. $C \subset$ obj $\hat{C}$ ~~[illegible]~~.

$(\tilde{S})_{S \in \text{obj}\, C}$ is generating family.

$S(T) = \text{Hom}(T, S)$. as usual.

$$G \hookrightarrow H$$

$$\begin{array}{ccc} \mathrm{Hom}(\tilde{S}, G) & \xrightarrow{\approx} & \mathrm{Hom}(\tilde{S}, H) \\ \| & \Downarrow & \| \\ \mathrm{Hom}(S, G) & G \xrightarrow{\sim} H. & \mathrm{Hom}(S, H) \\ \| & & \| \\ G(S) & & H(S). \end{array}$$

qed.

Giraud ~~Thm~~ E category with (1) – (4).

then (and only then) E is category of sheaves $\tilde{C}$ on some ~~category~~ site $\mathcal{C}$.

Topos – category of sheaves on a site

(topi)

just as good as category of sets.

$\boxed{\text{topology – study of topi.}}$

[Top = category of topological spaces]

X ~~$\Rightarrow$~~ $Top(X)$ (sheaves on X)

X, G if $G \cdot 1$ get $Top(X)$ back. $Top(X, G)$. sheaves on X where G operates compatible with operation on X.

G discreet group of automorphisms

$H^*(X, G, F)$

introduced in Tohoku paper.

if $X =$ pt. $\quad Top(\cdot, G) = B_G$

classifying topos.

(sheaf on pt = set.) $\quad$ sets on which G acts.

(plays role of classifying space, which is defined only up to homotopy).

$\Gamma(M) = M(e) = M^G$

derived functor = cohomology groups. $H^*(G, M)$.

unifies cohomology + topology.

E topos $\quad G$ group object in E

B_G : objects in E on which G acts.

Is topos again and plays role of classifying space.

Y covering $\mathscr{G}$

X_i

$X \supset U_\alpha$

Top category of all topological spaces.

if $\forall x \in X$, $\exists\, U_\alpha \ni x$ s.t. such that $\exists$ section over U_α some α.

See Lecture Notes Verdier Artin Grothendieck forthcoming.

can handle topological groups.

(Sch) = category of schemes.

FFQC topology on (Sch) (= FPQC) in French

faithfully flat quasi-compact

$$\begin{array}{c} S_i \\ \downarrow \\ S \end{array}$$

covering

want
① any ff.q.c morphism is covering
② covering in Zariski topology is covering.

we take coarsest topology with ① and ②

if $\exists$ $S_\alpha \subset S$, $S = \bigcup S_\alpha$, S_α affine

such that $\exists$ S'_α

$$\begin{array}{ccc} S_{i\alpha} & & S_i \\ \downarrow & & \downarrow \\ S_\alpha & \longrightarrow & S \end{array}$$

S_α fixed

$\exists$ ff.q.c morphism $S'_\alpha \to S_\alpha$ which factors through one of the $S_{i\alpha}$.

$$\begin{array}{ccc} S'_\alpha & \longrightarrow & S_{i(\alpha)} \\ & & \downarrow \\ & & S_\alpha \end{array}$$

(can take S'_α affine if desired)

$F : \text{Schemes}/_S \longrightarrow \text{Sets}.$

Is sheaf if

$$F(T) \to F(T_i) \rightrightarrows F(T_{ij})$$

exact for any covering family $\begin{matrix} T_i \\ \downarrow \\ T \end{matrix}$.

need check exactness only if (1) $T = \bigcup T_i$ (sheaf for Zariski topology)

(2) card $I = 1$. $T' \to T$ f.f. q.c. (over S).

can do for T, T' both affine

$X \qquad S \longmapsto \mathrm{Hom}(S, X).$

is sheaf in Zariski topology.

claim also is sheaf for f.f. q.c. topology.

$T \longmapsto \mathrm{Hom}(T, X) \qquad T' \to T$ f.f. q.c.

$T' \times_T T'$

$$F(T) \longrightarrow F(T') \rightrightarrows F(T' \times_T T') = F(T'')$$

$$T'' \rightrightarrows T' \xrightarrow[\text{ffqc.}]{} T$$

then
$$\underbrace{\mathrm{Hom}(T, X) \to \mathrm{Hom}(T', X) \rightrightarrows \mathrm{Hom}(T'', X)}_{\text{exact sequence.}}$$

$$T = T'/T''.$$

SGA I
Descent Theory
VIII

amounts to

$$A'' = A' \otimes_A A' \leftleftarrows A' \leftarrow A$$

(elts of A' where $\lambda \otimes 1 = 1 \otimes \lambda$ i.e. come from A)

<u>Deligne</u> - he invented stronger topology with just as many descent statements

ffqc covers inject ones

$$A \xrightarrow[\text{ffqc.}]{} \hat{A}$$

$$k \longrightarrow A$$
alg finitely generated over k.

March 18

$$\begin{tikzcd} \mathrm{Top}/S \arrow[r, hook, "\text{embedding of categories}"] & \mathrm{Top}/S \arrow[dl, "\text{something lost.}"] \\ \mathrm{Top}(S) & \end{tikzcd}$$

$$\begin{tikzcd} & S \\ \tilde{X} \arrow[ur] & \end{tikzcd}$$

Reference Topi etc. SGA 4 I – VI

IV exp.

Artin, Grothendieck, Verdier.

Will appear Springer Lecture $\boxed{\text{this year?}}$ Notes. (rewritten).

Eff Gr topology

— More coverings than in Zariski topology.

Zariski

$$\begin{array}{c} X_1 \\ \Big\downarrow \text{covering} \\ X \end{array}$$

if ~~∀~~ ∃ covering $X = \bigcup X_\alpha$ s.t. that ~~each~~ ∀α

$$\begin{tikzcd} X_1(\alpha) \arrow[d] & \\ X \arrow[r, hookleftarrow] & X_\alpha \arrow[ul, "\exists"'] \end{tikzcd}$$

zariski open.

$$\text{Schemes}/S \hookrightarrow \widetilde{\text{Schemes}/S} \hookrightarrow \widehat{\text{Schemes}/S}$$

each embedding full.

$$\begin{array}{l} G \\ \downarrow \\ S \end{array} \quad \text{group scheme}/S.$$

get B_G classifying topos.

S Schemes$/S$ big.

Take all
$$\begin{array}{ccc} X & \supset & V \\ \downarrow & & \downarrow \\ S & \supset & U = A \end{array}$$
$V = B$ schemes locally of finite presentation, U affine.

B is A-algebra of finite presentation.

finite presentation

$$\begin{array}{ccc} X_i & & \coprod X_i \\ {\scriptstyle f_i}\downarrow & \text{say is covering if} & \downarrow{\scriptstyle f.f.} \\ X & & X \end{array}$$

Coarser topology than f.f.q.c.

[ie any covering in above topology is covering in f.f.q.c.]

f_i flat, locally finite presentation
⇓
$f_i(X_i) \subset X$ is open.

f.f. f.p.

Etale topology. [induced by ffqc on category of étale schemes over ~~S~~ S]

$\tilde{S}_{et}$

needed to construct cohomology thing for Weil conjectures

$C(S)$ schemes locally of finite presentation over S.

$X \downarrow S$

$C(S) \hookrightarrow \widetilde{C(S)} \hookrightarrow \widehat{C(S)}$

don't lose anything.

$\mathcal{C}$ Site $F \xrightarrow{u} G$ $\tilde{\mathcal{C}}$

(i) u is mono in $\tilde{C}$

$\iff$ u is mono in $\hat{C}$

$\iff \forall S \in \mathcal{S},\ F(S) \to G(S)$ is injective.

No problem with mono. — argumentwise!

$\boxed{F \to G \text{ mono} \iff F \xrightarrow{\sim} F \times_G F \text{ so can be expressed in terms of inverse limits.}}$

$F \xrightarrow{u} G$ epi.

u epi in $\hat{C} \longleftrightarrow$ [$\forall S$ $u(S) : F(S) \to G(S)$ epi]

[$\tilde{C} \hookrightarrow \hat{C}$ does not commute with direct limits.
$\uparrow$ epi here $\uparrow$ not necess. here]

u epi in $\tilde{C} \iff \forall S \in$ obj $\mathcal{C}$, $g \in G(S)$ $\exists S_i \xrightarrow{\text{covering}} S$ such that $g | S_i \in \operatorname{Im}(F(S_i) \to G(S_i))$

"epi on stalks".

$$\begin{tikzcd} \mathcal{C} \arrow[r, "\varepsilon"] \arrow[d, "i"'] & \tilde{\mathcal{C}} \\ \hat{\mathcal{C}} \arrow[ur, "\text{associated sheaf}"'] & \end{tikzcd}$$

Looked at earlier where ε a chain

~~[illegible]~~

$$\begin{tikzcd} S_i \arrow[d] & \tilde{S}_i \arrow[d] \\ S & \tilde{S} = \varepsilon(S) \end{tikzcd}$$

covering $\Longleftrightarrow$ $\coprod \tilde{S}_i \xrightarrow{\text{epi}} \tilde{S}$

$\tilde{\mathcal{C}} \longrightarrow \hat{\mathcal{C}}$ not right exact.

$F \to G$ epi of sheaves

$\not\Downarrow$

$F(S) \to G(S)$ epi.

(site $\tilde{C}$

Passage to Quotient

Relation $R \hookrightarrow F \times F$
↑ subobject.

is equivalence relation $\Leftrightarrow$ $\forall S \in \text{obj}\, C$.

$R(S) \subset F(S) \times F(S)$ is equivalence relation.

$$R \rightrightarrows F \longrightarrow F/R.$$ quotient in category of sheaves.

$$R' \rightrightarrows F' \longrightarrow F'/R'$$

$R \times R' \qquad F \times F'$

$$F \times F' / R \times R' \cong F/R \times F'/R'.$$

the passage to quotient compatible with base change.

$R \qquad F \qquad F/R.$

equivalence relations in $F \times F$ which are coarser than R $\longleftrightarrow$ equivalence relations in $F/R \times F/R$.

$$\begin{tikzcd} R \arrow[r, shift left] \arrow[r, shift right] & X \arrow[r, dash] \arrow[d, dash] & X/R \\ & S & \end{tikzcd}$$

$R \subset X \times_S X$

$$\begin{tikzcd} A' \arrow[dr] & A \arrow[l, shift left] \arrow[l, shift right] \arrow[d, dash] & B \arrow[l] \arrow[ll, bend right] \\ & \mathcal{O}_S & \end{tikzcd}$$

biggest subsheaf where coincide.

Spec B = cokernel in category of schemes.
not sheaves.

This type of quotient not compatible with base change.

Defect in category of schemes

$\boxed{\text{Replace by category of sheaves to correct these defects.}}$

But sometimes not object which is not scheme.

C site

$\tilde{C}$ $\quad G$ grp object $\quad X$

$$G \times X \longrightarrow X$$
$$(g, x) \longmapsto gx$$

with sets: G operates freely if
$$gx = g'x \implies g = g'.$$

Say G operates freely on X if $G(S)$ acts freely on $X(S)$.

equivalent to say that

$$G \times X \xrightarrow{(\pi,\, pr_2)} X \times X$$
$$(g, x) \longmapsto (gx, x)$$

is mono.

~~Thus~~ Get $G \times X \hookrightarrow X \times X$

is equivalence relation.

$\therefore$ can form quotient object X/G.

If action not free can still do something similar, losing something.

$$G \times X \cong X \times_{X/G} X$$ in above case.

~~$F \longrightarrow G$~~

$$G \xrightarrow{u} H$$ group homomorphism of group objects in D.

G can act on H either by left or right mult. ($D = \mathcal{E}$ for example)

$$G \times H \longrightarrow H$$

$$(g,h) \longrightarrow u(g)\,h$$

or $$(g,h) \longrightarrow h\,u(g).$$

$G\backslash H$ left action $\qquad H/G$ right action $\qquad$ (of epistemology in $\mathcal{E}$.

If u a monomorphism in particular.

Have usual thms — (in sheaf case).

always in commutative case.

$$G \underset{\text{invariant}}{\hookrightarrow} H \qquad \left(\text{ie} \quad G(S) \triangleleft H(S) \quad \forall S\right)$$

Then
$$\left\{\begin{array}{l} H/G = G/H \\ \text{is a group object.} \qquad H/G \text{ quotient object.} \\ H \longrightarrow H/G \quad \text{group homomorphism.} \end{array}\right.$$

Assume
$$\begin{array}{ccc} G & \hookrightarrow & H \\ \cup & & \cup \\ G' & \hookrightarrow & H' \end{array} \qquad H/G' = \frac{G\,H'}{G}.$$

Representability of quotients

$$\underset{\substack{\text{scheme also} \\ \text{equivalence relation}}}{R} \underset{p_2}{\overset{p_1}{\rightrightarrows}} X \quad \text{scheme}$$

(gives equivalence relation in schemes in fpqc topology)

fpqc topology. (good for passage to quotient).

X/R = quotient sheaf.

Assume P_1, P_2 flat.

Is X/R a scheme?

Two ~~cases~~ cases

(a) P_1 finite and of finite presentation] => locally free.

ie $R = \operatorname{Spec} \mathcal{A}$
$\mathcal{A}$ locally free sheaf of algebras.

X/R is representable $\iff$ all orbits of R are contained in an affine set. (~~a~~ for each orbit $\exists$ affine open set containing it).

$P_2(P_1^{-1}(x)) = R(x)$ orbit of x. $\forall\, x \in X$.

Gives equivalence relation on underlying set of X.

eg if orbits finite, usually will be contained in affine.

eg X affine even.

$$R \overset{p_1}{\underset{p_2}{\rightrightarrows}} X \xrightarrow{p} Y \text{ scheme} \quad (= X/R)$$

locally free flat, finite, finite presentation

$[X]/[R] = Y$ (underlying sets)

(Y = underlying space of X/R if X/R represented by scheme)

$$Y \xleftarrow{p} X \leftleftarrows R$$

$$\mathcal{U} \hookleftarrow X_{\mathcal{U}} \leftleftarrows R_{\mathcal{U}}.$$

$$\mathcal{O}(\mathcal{U}) \longrightarrow \Gamma(X_{\mathcal{U}}, \mathcal{O}_{X_{\mathcal{U}}}) \rightrightarrows \Gamma(R_{\mathcal{U}}, \mathcal{O}_{R_{\mathcal{U}}}).$$

$\mathcal{O}$ gives sheaf of rings on Y. makes p morphism of ringed spaces.

$(Y, \mathcal{O})$ is a quotient of X/R in category of locally ringed spaces.

With above assumptions, represents quotient in sense of ~~schemes~~ sheaves.

(Artin - Renault?)

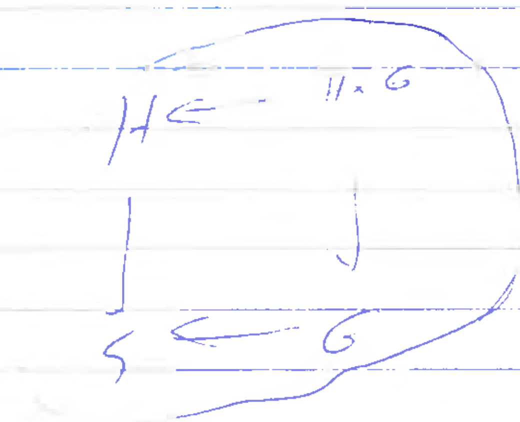
Artin's algebraic spaces used to cope with case where orbits not ~~contained in affine~~.

(b) $S = \operatorname{Spec} A$ $\qquad$ A artin ring (field even)

$H \xhookrightarrow{\text{closed subgroup}} G$ locally of finite type. scheme/S.

$$\begin{tikzcd} H \arrow[r, hook] \arrow[dr, "\text{flat}"'] & G \arrow[d] \\ & S \end{tikzcd}$$

Then $H \times G \rightrightarrows G$ flat
flat. (without artin S)



Then G/H is representable (also $H\backslash G$) by scheme locally of finite presentation

and $G \xrightarrow[\text{flat}]{\text{faithfully}} G/H$

If $S = \operatorname{Spec} \Lambda$, Λ dvr say
or dedekind

representable (Renault). [deeper]

If dim 2, counterexamples.

Sum up: In category of group schemes of finite over field — can take quotients and stay in same category.

March 19 Lawvere — feels category should be basic notion of mathematics.

$\mathcal{C}$ category with topology = site.

[eg S schemes with f.f.q.c. topology.

$$\mathcal{C} \hookrightarrow \tilde{\mathcal{C}} \hookrightarrow \hat{\mathcal{C}}$$

Can take quotient in $\tilde{\mathcal{C}}$

$$R \underset{p_2}{\overset{p_1}{\rightrightarrows}} X \xrightarrow{f} Y = X/R.$$

Assume Y representable. (ie quotient in $\mathcal{C}$.)
some criteria last time
with schemes.

Let P be a property of arrows in $\mathcal{C}$, of local type.

$$\begin{tikzcd} T & T' \arrow[l] \arrow[d, "f_i'"] \\ S \arrow[from=u, "f_i"'] & S' \arrow[l] \end{tikzcd}$$

① stable under base change. f_i have $P \Rightarrow f_i'$ have P.

② if they have the property locally they hold globally.

i.e. $S \leftarrow S_\alpha'$ cover. if $f'_{i\alpha}$ have the property $\forall \alpha$ then so does f_i.

Most properties in alg. geo. are of local type for f.f. q.c. topology.

~ back to $R \underset{p_2}{\overset{p_1}{\rightrightarrows}} X \xrightarrow{p} Y = X/R$.

If p_1, p_2 have P, so does p.

$$\begin{tikzcd} R \arrow[r, shift left, "p_1"] \arrow[r, shift right, "p_2"'] & X \arrow[r, "p"] & Y = X/R \\ & X \times_Y X \arrow[u, "p_1"] \arrow[r, "pr_2"'] & X \arrow[u, "p"] \end{tikzcd}$$

∴ p_1 deduced from p by base change pr_2.

Conversely, take base change by p, get $\longrightarrow$ p_1 satisfies $P \Leftrightarrow p$ satisfies P

~~could have quotient in $\mathcal{C}$, which is not quotient in $\tilde{\mathcal{C}}$~~

to say quotient in $\tilde{\mathcal{C}}$ is representable is stronger. Then is quotient sheaf and quotient scheme.

Third existence theorem for quotients

$$X \xrightarrow{f} Z$$

f locally of finite presentation.

$$R = X \times_Z X \underset{p_2}{\overset{p_1}{\rightrightarrows}} X$$

Assume that p_1 is flat ($\Rightarrow$ faithfully flat?)

X/R close to being representable.

~~If~~ have $X/R \xhookrightarrow{\text{mono.}} Z$ ~~Z representable.~~

Then X/R is representable.

(also enough to assume p_1 flat and locally of finite representation.)

~~pf. of~~ suffices enough to assume $\exists$ $X/R \xhookrightarrow{\text{mono.}} Z$ Z representable.

If X, Z are over S and if X is locally of finite presentation (type) over S.

then so is Y.

so quotient not too big.

of varieties → quotient is ...

$$\begin{tikzcd} Y' \arrow[r] \arrow[dr] & Y \arrow[d] \\ & S \end{tikzcd}$$

group schemes/S.

$$\begin{tikzcd} G \arrow[r, "u"] \arrow[dr] & H \arrow[d] \\ & S \end{tikzcd}$$

G, H locally of finite presentation over S.

assume $N = \ker u$ (group subscheme of G) is flat over S.

Then G/N (quotient in sense of sheaves. sheaf of groups) is representable.

Then u factors $G \xrightarrow[\text{(for sheaves)}]{\text{epi}} G/N \longrightarrow H$

(if S field — $\ker u$ always flat)

Pf Cor of above.

$$N \times G \underset{p_2}{\overset{p_1}{\rightrightarrows}} G \longrightarrow G/N.$$

$$\begin{tikzcd} N \arrow[d, "\text{flat}"'] & N \times_S G \arrow[d, "\text{flat}"] \\ S & G \arrow[l] \end{tikzcd} \quad \Longrightarrow$$

flat equivalence relation. embedded in representable functor (H.).

∴ is representable.

[fpqc topology]

Imp. schemes locally of finite presentation over field k.

$$\begin{tikzcd} G \arrow[rr, "u"] \arrow[dr] & & H \arrow[dl] \\ & k & \end{tikzcd}$$

G finite type

(1) Then u is monomorphism

$\Longleftrightarrow u$ is a closed immersion.

principle of proof By flat descent can assume k algebraically closed.

$\overline{u(G)} \subset H$

stable under translation by pts of G rational over k.

$\overline{u(G)} - u(G)$ smaller dim than $u(G)$.

dim $U(G)$ = dim G.

But if $p \in \overline{U(G)} - U(G)$ stable under translation.

But orbit $\cong G$. (G acts freely on H)

<u>Contradiction</u>.

<u>eg</u> If G not finite type.

$$G = \mathbb{Z}_K.$$

constant group scheme. group scheme/K

$$\mathbb{Z}_K \longrightarrow H$$

$1 \longmapsto$ elt of infinite order.

element of H of infinite order need not define closed ~~subscheme~~ subset.

$\therefore$ can't get closed immersion.

constructible

(2) U epimorphism of ~~schemes~~ sheaves in SF&C topology $\iff$ U is faithfully flat.

$\Rightarrow$

$N = \ker u \qquad H \cong G/N.$

$$\begin{tikzcd} & G/N \\ G \arrow[ur] & \end{tikzcd}$$

get u faithfully flat

e.g. if G, H affine

$G \xrightarrow{u} H$ epi

$\Updownarrow$

$B \longleftarrow A$ B is faithfully flat over A.

any morphism which admits section is covering

$$\begin{tikzcd} G \arrow[rr, "u"] \arrow[dr] & & H \arrow[dl] \\ & S & \end{tikzcd}$$

group schemes locally of finite presentation over S

1, u is mono $\Longleftrightarrow$ $\forall s \in S$, $u_s : G_s \longrightarrow H_s$ is monomorphism

(not special to group schemes)

(b) Assume G flat over S. (special to group schemes).

Then u is epi $\iff$ $\forall\ u_s : G_s \to H_s$ epi.

$\iff \forall s \in S$, ----- faithfully flat.

$\Longrightarrow H$ is flat over S.

$\Rightarrow$ clear since epi stable under base change.

Enough to prove: $\forall s \in S$ u_s f.f. $\Rightarrow$ u epi

$\Downarrow$

u flat

Can get faithful part by scheme argument.

by

$$\begin{tikzcd} X \arrow[rr, "u"] \arrow[dr] & & Y \arrow[dl] \\ & S & \end{tikzcd}$$

X, Y, locally finite presentation over S.

X flat over S. Then u flat $\iff$ $\forall s \in S$ $u_s : X_s \to Y_s$ flat.

(193)

$$\begin{array}{c} Y \\ U \text{ (open in } Y) \end{array}$$

$\Rightarrow$ $u(X)$ flat over S ~~flat over S~~ — if onto is faithfully flat.

Next talk — Thursday 2:30 on time.

Friday Usual time.

March 25

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d] & Y \arrow[dl] \\ S & \end{tikzcd}$$

X, Y locally of finite presentation over S.

Then f is monomorphism $\Longleftrightarrow$ $\forall s \in S$, $f_s : X_s \to Y_s$ is mono.

($\Rightarrow$ clear, must do $\Leftarrow$)

(1) We may assume $Y = S$.

?

(2) Reformulate

$$\begin{array}{c} Z \\ \big\downarrow \\ Y = \operatorname{Spec} k. \end{array} \quad \text{mono,} \quad \Rightarrow \quad Z = \operatorname{Spec} A \qquad \begin{array}{c} A \\ | \\ k. \end{array}$$

to be mono implies either that $A = 0$ empty scheme.
or $A \subsetneq K$.

$A \otimes A$ same. $1 \otimes x$ $x \otimes 1$

$\therefore A = k.$

2) <u>Reformulation</u> $\begin{array}{c} X \\ \downarrow \\ Y \end{array}$ mono $\Leftrightarrow$ { is identity or empty fibrewise }

Cover Y by affines Y_i.

$X \longrightarrow Y$ ~~aff~~ mono $\Leftrightarrow$ $X_i \longrightarrow Y_i$ affine.

3) We may assume $Y = \operatorname{Spec} A$.

X ~~quasi-compact~~. $= \varinjlim_i X_i$

and $X \longrightarrow Y$ mono $\Leftrightarrow$ $X_i \longrightarrow Y$ mono $\forall i$.

$\therefore$ may assume X quasi-compact.

4)

$$\begin{array}{c} X \\ | \\ A_i \hookrightarrow A \end{array}$$

$\uparrow$ finite presentation over integers

$A = \varinjlim A_i$

$$\begin{array}{ccc} X_i & & X \\ \downarrow & & | \\ A_i & \hookrightarrow & A \end{array}$$

X comes from X_i by base change if i big enough.

EGA IV - §9 constructibility justifies these considerations

$\therefore$ reduced to A noetherian

5) Lemma $U \xrightarrow{f} V$ schemes both locally of finite type.

S noetherian scheme.

(-A 3 ? expose XI ; construction techniques)

Then f is isomorphism $\Leftrightarrow$ $\forall s \in S$, and all $n \geq 1$.

$f_n : U \times_S S_n \longrightarrow V \times_S S_n$ is an isomorphism.

$$S_n = \mathrm{Spec}\left(\mathcal{O}_{S,s} / m_s^n\right)$$

<u>Corollary</u> f monomorphism

$\Longleftrightarrow$ $\forall s, n$ as before $f_n : X \times_S S_n \to Y \times_S S_n$ is monomorphism.

(<u>since</u> $U \times_V U \xleftarrow{\Delta_U} U$

f is monomorphism $\Longleftrightarrow$ Δ_U is isomorphism

Base change commutes with fibered product?)

6)
$$\begin{array}{ccc} X & \longrightarrow & Y \\ \uparrow & & \Big\downarrow \text{noetherian} \\ X' & \longrightarrow & Y_1' \end{array}$$

fiberwise mono $\Rightarrow$ monomorphism

local artinian

we may assume that Y is local and artinian. $Y = \mathrm{Spec}\, A$ A local artinian.

$X = \operatorname{Spec} B$ $\qquad$ B algebra $/A$.

$$\begin{tikzcd} B \arrow[r] \arrow[d, dash] & B_0 = B \otimes_A K = B/mB \\ m \subset A \arrow[r, dash] & K \arrow[u, "\text{epi}"'] \end{tikzcd}$$

$K \leftarrow$ residue class field.

$\therefore$ $\begin{matrix} B \\ \uparrow \\ A \end{matrix}$ epi (Nakayama).

$\begin{matrix} \operatorname{Spec} B \\ \downarrow \\ \operatorname{Spec} A \end{matrix}$ closed immersion $\Longrightarrow$ monomorphism.

$X = X' \amalg X''$ $\qquad$ $X' = (Y - y)$ $\quad$ $X'' = y$.

Examples

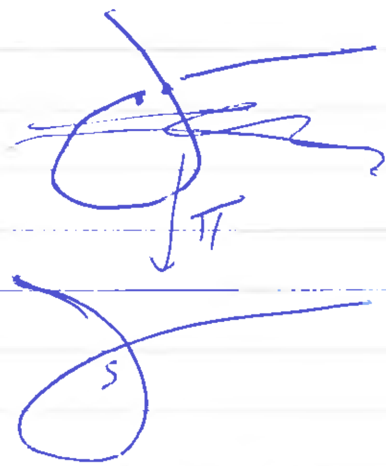
Y affine line.

$$\begin{matrix} X' \amalg X'' \\ \downarrow \pi \\ Y \end{matrix}$$

is monomorphism, but not immersion.

$$\begin{array}{c} X \text{ irreducible} \\ \Big\downarrow f \\ Y \text{ irreducible} \end{array}$$

f dominating, f mono.



(Y normal $\Rightarrow$ f open immersion ZMT)

Take $Y =$

remove one of pts in normalization of S.

Then get monomorphism. Clearly not isomorphism. Formally is immersion.

Mono of graph schemes / field is closed immersion
Arbitrary base change — need not be immersion.

$$1 \longrightarrow G' \xrightarrow{f} G \longrightarrow G'' \longrightarrow 1$$

exact if f mono and G'' is quotient G/G'.

Examples

Kummer sequences

$$0 \longrightarrow \mu_n \longrightarrow \mathbb{G}_m \xrightarrow{\lambda \mapsto \lambda^n} \mathbb{G}_m \longrightarrow 0 .$$

need check only epi here (in some topology).

S $\quad f \in \mathcal{O}_S^*$

$f = x^n$?

check can do locally in flat topology

$\mathcal{O}_S[X]/(X^n - f)$, sheaf of algebras over S, free of rank n.

$S' \longrightarrow S$ fl.

(is étale even, if throw out primes that divide n)
eg if working over field char 0)
then is epi in étale topology

eg 2. Artin - Schreier

char p

$$x \xrightarrow[\text{in char } p.]{\text{homo.}^{\text{ft}} \text{ since}} x^p - x$$

$$0 \longrightarrow (\mathbb{Z}/p\mathbb{Z})_{\mathbb{F}_p} \longrightarrow \mathbb{G}_a \longrightarrow \mathbb{G}_a \longrightarrow 0$$

just restricted over $\mathbb{F}_p$.

is epi

$f \in \Gamma(S, \mathcal{O}_S)$

want $f = x^p - x$.

Make base change (fppf topology)

$$\frac{\mathcal{O}_S[X]}{(X^p - X - f)}$$

free of rank p over $\mathcal{O}_S$.

Even epi for étale topology.

i.e. can do artin schreier theory for galois coverings of $\mathbb{G}_a$?

eg 3

$$0 \to (\mathbb{G}_m)_S \to GL(n)_S \xrightarrow{\text{faithfully flat}} GP(n-1)_S \to 1$$

diagonal matrices with same entry

fppf topology / Zariski topology

theorem.

$$GL(n)_S = \mathrm{Aut}_{\mathcal{O}_S}(\mathcal{O}_S^n)$$

$$GP(n-1)_S = \mathrm{Aut}(\mathbb{P}_S^{n-1}).$$

$$1 \to \mathbb{G}_m \longrightarrow GL(E) \longrightarrow PGL(E) \to 1$$

$$\begin{array}{c} \| \\ \mathrm{Aut}(P(E)) \\ \text{autos of } P(E) \end{array}$$

$$f \circlearrowright X = P(E)$$

$\mathcal{O}_X(1)$ — locally free rank 1 quotient.

$$E_s' \longrightarrow L$$

$$E_X \longrightarrow \mathcal{O}_X(1) \to 0$$

$$\pi: X \to S \longleftarrow S'$$

$$\underset{\substack{\| \\ \zeta}}{f^*(\mathcal{O}_X(1))} \cong \underset{\substack{\text{invertible} \\ \text{on } S}}{f^*\mathcal{M}} \otimes \mathcal{O}_X(m).$$

m positive or negative.

Since autos. $\mathcal{M} = 1$, $\mathcal{M}$ locally = structure sheaf.

not canonically once $\mathcal{M}$ sent equal to structure sheaf.

locally on S. $\zeta \cong \mathcal{O}_X(1)$ such iso. given by scalar.

indeterminacy in lifting comes here.

Deduce new $E = \pi_*(\zeta) \xrightarrow{\sim} \pi_*(\mathcal{O}_X(1)) \xleftarrow[\text{Canon.}]{\sim} E$

$$\pi_*(f^*\mathcal{O}_X(1)) = \pi_*(\mathcal{O}_X(1)).$$

get section of $GL(E)$ which lifts $GP(E)$.

(Try field case — auto. of $\mathbb{P}^{n-1}_k$ given by general linear group)

eg 4

$$1 \to SL(n) \to GL(n) \xrightarrow{\det} G_m \to 0$$

$\curvearrowright SL(E)$ $\quad$ $\curvearrowright GL(E)$ E locally free

epi — since $\exists$ multiplicative section

$$\begin{pmatrix} \lambda & & 0 \\ & 1 & \\ 0 & & 1 \end{pmatrix} \longleftarrow \lambda$$

(if E arbitrary locally free sheaf may not be global section, but will be locally (Zariski-topology).

References

March 26 Mumford Red book

Mumford Orange book. Construction of Picard Variety of Surface

give motivation for

EGA reference book.

SGA SGA III Group Schemes. (best to just look things up).

Fondements de la Geometrie Algebrique $\boxed{\text{10 Bourbaki talks}}$

Institut Henri Poincaré.

Serre Groupes Algébrique et Corps de classe. $\boxed{\text{geometric}}$

(Jacobian is extension of abelian variety by linear variety). is singular curve

$\boxed{\text{Algebraic Groups}}$

Gabriel Demazure – includes 150 page introduction to algebraic geometry.

North Holland.

Borel - Serre. Le théorème de Riemann-Roch.

short! 1958

SGA I Preliminaries on schemes.
Fundamental Group. (of curve of genus g with pts removed)
(reduce to char 0 then use transcendental groups)

X curve n.s. connected compact.

$$\downarrow$$

k alg. closed field.

$g =$ genus
$= \dim H^0(X, \Omega^1_X)$
$= \dim H^1(X, \mathcal{O}_X)$.

(Topological) Fundamental group has generators
Free on $x_1, \dots, x_{2g}$ generators,
with defining $(x_1, y_1) \cdots (x_g, y_g) = 1$

$(x_1, y_1) =$ commutator

with n pts removed has
$x_1, \dots x_{2g}$ $z_1, \dots z_n$ as generators
and relation $(x_1, y_1) \cdots (x_g, y_g) z_1 \cdots z_n = 1$

or free on $x_1, \dots x_{2g}, z_1, \dots z_{n-1}$.

Purely transcendental proof.

Then take ~~profinite completion~~ to get algebraic fundamental group. (No algebraic proof of algebraic fundamental group known).

Except $\mathbb{P}^1$

$$X = \mathbb{P}^1_{\mathbb{C}} \quad \text{remove } x, y, z.$$

Galois covering: group leaving x', y', z' resp. fixed

Sys x', y', z' (except ramifying at x etc)

$$X_1 \quad G \supset G_{x'}, \quad G_{y'} \quad G_{z'}.$$

finite; ramified only at x, y, z.

$$\big\downarrow$$

$$X \qquad (x, y, z)$$

z_1, z_2, z_3. cyclic of order equal to order of ramification ~~of ramification~~ ~~point~~.

Fundamental group then says can choose G_1, G_2, G_3 so that G generated by z_1, z_2, z_3 with relation $z_1 z_2 z_3 = 1$.

This then cannot be proved algebraically.

Also $\exists$ coverings of certain types.

Fundamental group does not change if you make extension of alg. closed field to another. (SGA ...).

that discussed as well.

Also theory of flat descent.
(can descend schemes as well
as properties).

Descent very important in
algebraic geometry.
Analogue to topologically
giving fibre space by giving over
open subsets, with isomorphisms on
intersections
top. not used.

Will soon appear as Springer Lecture Notes.

SGA2 Local and Global Lefschetz Theorem.
(technical)

most impt. part. $\boxed{\text{Local cohomology.}}$

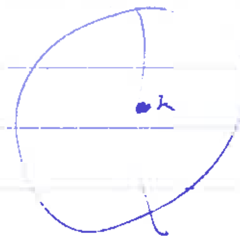
$X \supset Y$ compares fundamental group

projective hyperplane section and picard groups of X and Y.

X = local scheme — point

Y = $(f) - 0$

compare schemes X and Y
sometimes iso.



x removed from X and Y.

Local and global theorems interact.

Conjecture of Samuel proved

more or less { A normal local, dim ≥ 4, complete intersection ~~[illegible]~~
then is UFD.

No other proof known.

Some thms. of SGA 2 used in etale cohomology.
(SGAD)

SGA3 Seminar on group schemes. huge.

Theorems on representing functors.

(Gabriel Demazure covers only small part of seminar — and additional topics — Witt ~~group~~ vectors, Dieudonné modules.

Witt vectors in ~~[illegible]~~

(SGAA)

SGA 4 Theory of Topoi + Etale Cohomology.

the Basic theorems

that torsion coefficients

base change thus.

Rewritten.

Will appear in Springer Lecture Notes.

3 volumes.

$\boxed{\text{Best for } \underline{\text{general} } \text{ topology.} }$

just continuation of SGA 4

SGA 5 ℓ-adic cohomology.

(pass to limit) much technical work needed.

(Jouanolou). improved the thesis.

local duality. Lefschetz formula.

cohomology classes associated to cycles.

Application to L-functions.

chern classes { blown up varieties

flag bundles ...

will come out ~~in~~ end of year.

maybe.

two parts (Deligne delayed after Grothendieck quit — part still appears in Springer)

SGA 7 Monodromy groups.

deeper theorems than 4, 5 more geometry.
more specialist nature / algebraic variety case only.

$$\begin{tikzcd} X \arrow[d, "f"'] \\ Y \end{tikzcd}$$

Z_ℓ sheaf structures

fibering: topology or alg. geo.

$$R^i f_* (Z_\ell).$$

if f proper with non-singular fibres is local system.

Thm. Y = Spec dvr, alg. closed r.f. minus origin.

(some

$T \in \hat{Z}$. Tate unipotently on $H^i(X_{\bar{z}}, Z_\ell)$

"vanishing cycles". (as Lefschetz)
analysis depends on type of singularity.

Picard-Lefschetz formula.

SGA 6 Riemann Roch Theorem and Intersection Theory.

Outcome of Borel-Serre paper.
(Appendix to Hirzebruch however relation of two types)

G. arb. characteristic
H. transcendental — only for $\mathbb{C}$.

$$\begin{array}{c} X \\ \downarrow \\ Y \end{array}$$

G's thm: arb. proper morphism. Locally of complete intersection (maybe)

H's thm: only until Y = point. H has nice topological proof } should not need for algebraic geometry. Maybe for topology.

read Borel - Serre first !!!

Triangulated categories — Verdier. (technique used in SGA 4 even.)

Read Introduction to SGA 6. (discusses very concrete open problems).

Alg. Intersection Theory

X ns. alg variety over alg closed field

$$X \supset Z \quad (d)$$

$$d' \quad Z' \qquad\qquad Z \cap Z' \text{ defined}$$

Serre — Multiplicité (Algèbre locale et multiplicités) Springer. Define multiplicities of intersection.

G introduces (rational, algebraic) equivalence relation on cycles. (Form chow ring $A(X)$)

↳ defined in some seminar of Chevalley. (not done properly — proof of moving lemma false but can be corrected.)

$$A(X) \xrightarrow{\approx} gr\, K(X)$$

up to torsion (coherent sheaves)

(suggested already in Serre's treatment of multiplicities).

$A(X)$ makes sense only in ns. quasi projective case.

But $K(X)$ works anywhere (SGA 6). Think of $Gr\, K(X)$ as the chow ring — then can reformulate R.R.

Nice to redefine $A(X)$ for X noetherian, regular, and with invertible ample sheaf. (G knows how to def — but moving lemma is problem)

<u>Nice Case</u> Punctured spec of local ring (Regular outside max. ideal).

Chow ring $A(X)$ of such would be interesting (not needed here). Concrete geometric problem.

$\boxed{\text{SGA6 not needed here}}$

Artin - Mazur étale homotopy theory uses SGA 4. (not 5).

"pro-simplicial sets".