Letter to H. Hironaka
6.7.1962

par
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Transcription by

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How to cite:

A. Grothendieck. *Letter to H. Hironaka*. Unpublished letter, 6.7.1962. Transcription by M. Carmona et al., CSG, Grothendieck Institute. Draft, April 2024.

Neuilly July 6 1962

Dear Hironaka,

I had a little thought over our conversation last tuesday, it occurred to me that the type of argument I used yields in fact the following stronger result:

Theorem. — *Let* $f : X \longrightarrow Y$ *be a proper morphism of analytic spaces over* $\mathbf{C}$, *let* $y \in Y$, $Y_n = \mathit{Specan}(\mathscr{O}_y/\underline{m}_y^{n+1})$, $X_n = X \times_Y Y_n$,

$$\mathrm{Pic}(X_y) = \mathrm{R}^1 f(\mathscr{O}_X^\bullet)_y = \varinjlim \mathrm{Pic}(f^{-1}(U)) \quad y \in U,$$

$$\mathrm{Pic}(\hat{X}_y) = \varprojlim \mathrm{Pic}(X_n),$$

and consider the canonical homomorphisms

$$\mathrm{Pic}(X_y) \xrightarrow{u} \mathrm{Pic}(\hat{X}_y) \xrightarrow{v_n} \mathrm{Pic}(X_n)$$

Then the following are true:

(i) *The inverse system* $(\mathrm{Pic}(X_n))_{n\in\mathbf{N}}$ *satisfies the condition of Mittag-Leffler (even with Artin-Riesz type of uniformity).*

(ii) $\mathrm{Im} v_n = \mathrm{Im} v_n u =$ *(in virtue of (i)) set of universal images of* $\mathrm{Pic}(X_n)$ *in the inverse system* $(\mathrm{Pic}(X_m))_{m\in\mathbf{N}}$.

(iii) *In order for u to be an isomorphism, it is necessary and sufficient that* $\mathrm{R}^1 f_*(\mathscr{O}_X)$ *has a support not containing y or having y as an isolated point, i.e.* $\mathrm{R}^1 f_*(\mathscr{O}_X)_y$ *is a module of finite length.*

In fact (i) can be more precise:

(ibis) *In the inverse system* $(\underline{\mathrm{Pic}}_{X_n/\mathbf{C}})_{n\in\mathbf{N}}$ *of analytic groups, the system of the "Néron-Séveri groups" is constant for n large, whereas for* $m > n$ *and n large, the Kernel and Cokernel of*

$$\underline{\mathrm{Pic}}_{X_m/\mathbf{C}} \longrightarrow \underline{\mathrm{Pic}}_{X_n/\mathbf{C}}$$

are just vector groups.

Parts (i) and (ii) yield the

Corollary **1**. — *The following conditions are equivalent:*

(i) There exists an open $U y$ such that $X U$ is projective over U

(ii) For every n, X_n is a projective analytic space.

For instance, if $\dim X_0 \leq 1$, then (ii) and hence (i) holds.

The proof of the theorem only uses Grauert's analogues of the algebraic theorems of finiteness and comparison for direct images (of his blue paper) and the usual exact sequences $0 \longrightarrow \mathscr{Z} \longrightarrow \mathscr{O} \longrightarrow \mathscr{O}^{\bullet} \longrightarrow 0$, together with some standard use of Mittag-Leffler story and five lemma. It is valid in fact for any $\mathrm{H}^i(\mathscr{O}^{\circ})$, not only $i = 1$ (which seems the only one however to have geometric significance).

In the case of a formal scheme proper over a complete noetherian local ring with residue field of *characteristic* 0, the analog of the previous theorem (reducing to statements (i), (i bis)) hold true, and I wrote a purely algebraic proof of this, relying only on the fact that the kernel and Cokernel of $\underline{\mathrm{Pic}}_{X_{n+1}} \longrightarrow \underline{\mathrm{Pic}}_{X_n}$ are without torsion, and Néron's finiteness theorem; in particular, the analog of corollary 1 holds true in this case. These results break down of course in car. > 0.

However, using the (as yet unwritten !) Gaga of Serre-Grauert-Remmert-Grothendieck (of Grauert-Remmert's paper, complemented by the method of an old talk of mine in Cartan's Seminar, to recover the case of proper morphisms of schemes from the projective one, via Chow's lemma...), the analytic theorem above yields an interesting intrinsic property of analytic algebras over **C**, with respect to algebraic geometry over such a local ring:

Theorem. — *Let A be an analytic algebra over* **C** *(we can suppose A to be the ring of convergent power series in n variables), $\hat{A}$ its completion, Y and $\hat{Y}$ the spectra, X a proper scheme over Y, $\hat{X} = X \times_Y \hat{Y}$. Then*

(i) The inverse system $(\mathrm{Pic}(X_n))$ *satisfies MLAR (as stated above, this depends only on the car. 0 assumption for the residue field).*

(ii) $\mathrm{Pic}(X)$ *and* $\mathrm{Pic}(\hat{X})$ *have same image in* $\mathrm{Pic}(X_n)$, *– namely the group of "universal images". (NB recall* $\mathrm{Pic}(\hat{X}) \simeq \varprojlim \mathrm{Pic}(X_n)$*).*

(iii) *In order for* $\mathrm{Pic}(X) \longrightarrow \mathrm{Pic}(\hat{X})$ *to be an isomorphism, it is necessary and sufficient that* $\mathrm{supp}\, \mathrm{R}^1 f_*(\mathcal{O}_X) \subset (y)$, *i.e.* $\mathrm{H}^1(X, \mathcal{O}_X)$ *of finite length over A. (NB It amounts also to the same to ask that the inverse system of the subgroups* $\mathrm{Pic}'(X_n)$ *of universal images is constant for large n).*

We get for example:

Corollary **1**. — *The following conditions are equivalent:*

(i) X/Y *projective*

(ii) $\hat{X}/\hat{Y}$ *projective*

(iii) *For every* n, X_n/Y_n *projective.*

This applies for instance if $\dim X_0 \leq 1$.

Now applying your theorem of *resolution of singularities*, and Mumford's method of relating the local Picard group of A to the global Picard group of a regular scheme dominating A birationally, one gets from the last statement (iii) of last theorem:

Corollary **2**. — *Let A be as above, assume* $Y' = Y - (y)$ *regular, and let* $\hat{Y}' = \hat{Y} - (\hat{y})$. *Then* $\mathrm{Pic}(Y') \longrightarrow \mathrm{Pic}(\hat{Y}')$ *is an isomorphism.*

This explains "à priori" (when A is normal) why Mumford was able to introduce on the group of divisor-classes of A a structure of an analytic group (which in fact is algebraic…), which from the algebraic point of view should be possible rather for the group of divisors classes of the completion $\hat{A}$; of course Mumford uses directly the same kind of argument I used.

I do no know if in the last statement, the hypothesis that Y' is regular ("y isolated singularity") is essential; we could dispense with it and replace it by "A reduced" if you can prove by your theory of resolution the following: if $f : X \longrightarrow Y$ is proper "birational", X regular, then $\mathrm{R}^1 f_*(\mathcal{O}_X) = 0$. I understand you prove this if Y also is regular (which is easily checked by your theory), but I wonder if this is really needed. I would not be surprised either if in this statement, Y' can be replaced by any open subset of Y (replacing of course $\hat{Y}'$ by the inverse image of the latter). Moreover, I would expect the analogous statements to hold for π_1, more generally for all "topological" invariants as Weil homology, homotopy groups etc, that can be defined for schemes. This should be related to the fact that all these invariants vanish for the geometric fibers of the morphism $\mathrm{Spec}(\hat{A}) \longrightarrow \mathrm{Spec}(A)$. This is easy to check at least for π_0 (and is true in fact for any henselian ring which is a "good" ring); however I do not know if this is true also for π_1.

Besides, I would not be surprised if most of the previous results (namely parts (ii) and (iii) of the second theorem, and the two corollaries, as well as the conjectures of the previous sections) did hold true for any "good" ring which is henselian or at least for the "henselian closures" of the local rings arising from algebras of finite type over a field, or over the integers, - although I do not have any result along these lines (except those stemming from my remark on $\overline{k}_0$). This can be stated of course directly in terms of conjectures for the latter local rings without explicit reference to a henselian closure, for instance corollary 2 would yield the conjectural statement: Let A be a local ring of an algebra of finite type over a field, $\hat{A}$ its completion, Y and $\hat{Y}$ the spectra, Y' and $\hat{Y}'$ the complements of the closed points, then any invertible sheaf on $\hat{Y}'$ can be defined by an invertible sheaf on some Y_1', where Y_1 is local and $Y_1 \longrightarrow Y$ is étale with trivial residue field extension (i.e. inducing an isomorphism for the completions $A \longrightarrow A_1$). I wonder what information is given by Mumford's example in his blue paper, p.16, which I believe yields a case where the invertible sheaf considered does not come from an invertible sheaf on Y'? I was not able to understand his construction.

Anyhow, one should be able to determine whether or not the analytic algebras over $\mathbf{C}$ have any significant intrinsic property which is not shared by all "good" henselian rings with residue field of car. 0 (I recall that by good I mean "quotient of a regular local ring B such that the fibers of $\mathrm{Spec}\,\hat{B} \longrightarrow \mathrm{Spec}(B)$ are universally regular).

Please give my regards to Waka, and also Mireille's; she just got the parcel from Waka, and was extremely pleased, in fact, she slipped into her new bed-shirt on the spot, and is delighted by it in every respect.

Sincerely yours