

Letter to J. Coates

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par

Alexander Grothendieck

Transcription by



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Dear Coates,

Here a few more comments to my talk on the conjectures. The following proposition shows that the conjecture $C_\ell(X)$ is independent of the chosen polarization, and has also some extra interest, in showing the part played by the fact that $H^i(X)$ should be “motive-theoretically” isomorphic to its natural dual $H^{2n-i}(X)$ (as usual, I drop the twist for simplicity).

Proposition. — *The condition $C_\ell(X)$ is equivalent also to each of the following conditions:*

- a) *$D_\ell(X)$ holds, and for every $i < n$, there exists an isomorphism $H^{2n-i}(X) \longrightarrow H^i(X)$ which is algebraic (i.e. induced by an algebraic correspondence class; we do not make any assertion on what it induces in degrees different from $2n - i$).*
- b) *For every endomorphism $H^i(X) \longrightarrow H^i(X)$ which is algebraic, the coefficients of the characteristic polynomial are rational, and for every $i < n$, there exists an isomorphism $H^{2n-i}(X) \longrightarrow H^i(X)$ which is algebraic.*

Proof. I sketched already how $D_\ell(X)$ implies the fact that for an algebraic endomorphism of $H^i(X)$, the coefficients of the characteristic polynomial are rational numbers, therefore we know that a) implies b), and of course $C_\ell(X)$ implies a). It remains to prove that b) implies $C_\ell(X)$. Let $u : H^{2n-i}(X) \longrightarrow H^i(X)$ be the given isomorphism which is algebraic, and $v : H^i(X) \longrightarrow H^{2n-i}(X)$ is an algebraic isomorphism in the opposite direction, induced by L_X^{n-i} . Then $uv = w$ is an automorphism of $H^i(X)$ which is algebraic, and the Hamilton-Cayley formula $u^b - \sigma_1(w)u^{b-1} + \dots + (-1)^b \sigma_b(w) = 0$ (where the $\sigma_i(w)$ are the coefficients of the characteristic polynomial of w) such that w^{-1} is a linear combination of the w^i , with coefficients of the type $+/- \sigma_i(w)/\sigma_b(w)$ (N.B. $b = \text{rank } H^i$). The assumption implies that these coefficients are rational, which implies that w^{-1} is algebraic, and so is $w^{-1}u = v^{-1}$, which was to be proved.

N.B. In characteristic 0, the statement simplifies to: $C(X)$ equivalent to the existence of algebraic isomorphisms $H^{2n-i}(X) \longrightarrow H^i(X)$, (as the preliminary in

b) is then automatically satisfied). Maybe with some extra care this can be proved too in arbitrary characteristics.

Corollary. — *Assume X and X' satisfy condition C_ℓ , and let $u : H^i(X) \longrightarrow H^{i+2D}(X') \longrightarrow H^i(X)$ ($D \in \mathbf{Z}$) be an isomorphism which is algebraic. Then u^{-1} is algebraic.*

Indeed, the two spaces can be identified “algebraically” (both directions!) to their dual, so that the transpose of u can be viewed as an isomorphism $u' : H^{i+2D}(X') \longrightarrow H^i(X)$. Thus $u'u$ is an algebraic automorphism w of $H^i(X)$, and by the previous argument we see that w^{-1} is algebraic, hence so is $u^{-1} = w^{-1}u'$.

As a consequence, we see that if $x \in H^i(X)$ is such that $u(x)$ is algebraic (i being now assumed to be even), then so is x . The same result should hold in fact if u is a monomorphism, the reason being that in this case there should exist a left-inverse which is algebraic; this exists indeed in a case like $H^{n-1}(X) \longrightarrow H^{n-1}(Y)$ (where we take the left inverse $\Lambda_X \varphi_*$). But to get it in general, it seems we need moreover the Hodge index relation. (The complete yoga then being that we have the category of motives which is semi-simple!). Without speaking of motives, and staying down on earth, it would be nice to explain in the notes that $C(X)$ together with the index relation $I(X \times X)$ implies that the ring of correspondences classes for X is semi-simple, and how one deduces from this the existence of left and right inverses as looked for above.

This could be given in an extra paragraph (which I did not really touch upon in the talk), containing also the deduction of the Weil conjectures from the conjectures C and A .

A last and rather trivial remark is the following. Let's introduce variant $A'_\ell(X)$ and $A''_\ell(X)$ as follows:

$A'_\ell(X)$: if $2i \leq n - 1$, any element x of $H^i(X)$ whose image in $H^i(Y)$ is algebraic, is algebraic.

$A''_\ell(X)$: if $2i \geq n - 1$, any algebraic element of $H^{i+2}(X)$ is the image of an algebraic element of $H^i(Y)$.

Let us consider also the specifications $A'_\ell(X)^\circ$ and $A''_\ell(X)^\circ$, where we restrict to the critical dimensions $2i = n - 1$ if n odd, $2i = n - 2$ if n even. All these conditions are in the nature of “weak” Lefschetz relations, and they are trivially implied by

$A_\ell(X)$ resp. $C_\ell(X)$ (in the first case, applying φ we see that $L_X X$ is algebraic; in the second, we take $y = \Lambda_Y \varphi^+(x)$). The remark then is that these pretentively “weak” variants in fact imply the full Lefschetz relations for algebraic cycles, namely:

Proposition. — $C_\ell(X)$ is equivalent to the conjunction $C_\ell(Y) + A_\ell(X \times X)^\circ + A''_\ell(X \times X)^\circ$, hence (by induction) also to the conjunction of the conditions A'_ℓ and A''_ℓ for all of the varieties $X \times X, Y \times Y, Z \times Z, \dots$. Analogous statement with $X \times Y, Y \times Z$ etc instead of $X \times X, Y \times Y$ etc.

This comes from the remark that $A_\ell(X)^\circ$ follows from the conjunction of $A'_\ell(X)^\circ$ and $A''_\ell(X)^\circ$, as one sees by decomposing $L_X^2 : H^{2m-2}(X) \longrightarrow H^{2m+2}(X)$ into $H^{2m+2}(X) \xrightarrow{\varphi^k} H^{2m+2}(Y) \xrightarrow{\varphi_\alpha} H^{2m}(X) \xrightarrow{L_X} H^{2m+2}(X)$ if $\dim X = 2m$ is even, and $H^{2m+1-1}(X) \longrightarrow H^{2m+1+1}$ into $H^{2m}(X) \xrightarrow{\varphi^*} H^{2m}(Y) \xrightarrow{\varphi_*} H^{2m+2}(X)$ if $\dim X = 2m + 1$ is odd.

Sincerely yours

