CLASSIFYING TOPOS

by

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The basic facts about the classifying topos of a stack of groupoids were first stated in [3] and are exposed in detail in [4] Ch. VIII. This construction is useful in cohomology theory and has been introduced independently by D. Mumford to study the moduli of elliptic curves [7]. Algebraic stacks of groupoids are used in algebraic geometry cf. [1]. Here a simpler and more general approach allows us to treat the case of a stack whose fibers are not supposed to be groupoids. As a by-product we get the existence of fibered products in the bicategory of topos. This result was first announced by M. Hakim several years ago but was never published. I suspect that any written proof would have to deal with rather subtle technical difficulties about finite limits which are overcome here by the results of §1 .

If $\underline{S}$ is a site we use the work stack for the french champ [4] and prestack for prechamp (a prestack is merely a fibered category over the underlying category of the site) and split stack for champ scindé. Up to equivalence a split stack can be viewed as a sheaf of categories over $\underline{S}$ (or a category-object of the corresponding topos) satisfying some extra condition namely the patching of objects. As usual we choose and fix a universe $\underline{U}$. For clarity it should be recalled that a $\underline{U}$-topos is a special case of $\underline{U}$-site [5] and that any category can be viewed as a site such that any presheaf is a sheaf and any prestack is a stack.

§1. Left exact stacks.

A category is left exact if it admits finite limits. A functor $f:A \longrightarrow B$ between left exact categories A and B is left exact if it preserves finite limits. A site is said to be left exact if the

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underlying category is so. A stack C over a site $\underline{S}$ is said to be left exact if its fibers are left exact and if for any map $f:T \to S$ in $\underline{S}$ the inverse image functor induced by f between the fibers of C is left exact.

Lemma 1.1. A stack C over a left exact site $\underline{S}$ is left exact if and only if the underlying category and the structural functor $p:C \to \underline{S}$ are left exact.

The proof rests on the fact that a commutative square of C whose projection is cartesian in $\underline{S}$ is cartesian as soon as two opposite sides are $\underline{S}$-cartesian.

Lemma 1.2. A morphism $m:A \to B$ between two left exact stacks is left exact if and only if for any $S \in |\underline{S}|$[1] the functor $m_S:A_S \to B_S$ induced by m between the fibers at S is left exact.

Proposition 1.3. Let $f:\underline{S}' \longrightarrow \underline{S}$ be a morphism between two sites (e.g. two topos) . Then the direct image (resp. inverse image) of a left exact stack and of a left exact morphism of stacks over $\underline{S}'$ (resp. $\underline{S}$) is left exact.

1.3.1. The direct image of a stack being nothing but pull-back along the underlying functor $f^*:\underline{S} \longrightarrow \underline{S}'$ of f , preserves the fibers, hence the left exactness. To treat the case of the inverse image by f of a stack over $\underline{S}$ we will use the following caracterisation of left-exactness.

1.3.2. First let I be a finite category. For any stack F over $\underline{S}$ let F^I be the prestack whose fiber at $S \in |\underline{S}|$ is the category of functors from I to the fiber F_S . One checks easily that this is a stack provided with a morphism of stacks (constant diagrams)

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(1) The set of objects of a category C is denoted by $|C|$.

(1) $$cF : F \longrightarrow F^{I} \quad .$$

Furthermore F <u>is left exact if and only if for any finite category</u> I cF <u>admits a right adjoint in the bicategory of stacks</u>. The if part is obvious since such an adjoint λ induces an adjoint to each functor cF_S , $S \varepsilon |\underline{S}|$, induced by cF on the fibers at S and since λ is cartesian. The only if part is no more difficult than (1.2). Since the property of having a right adjoint is preserved by morphisms of bicategories and since the inverse image of stacks is such a morphism [4] p.88, it remains to show the following.

<u>Lemma 1.3.3</u>. One has a natural equivalence $e : f^*(F^I) \longrightarrow f^*(F)^I$ such that $e.f^*(cF) = cf^*(F)$.

According to [4] p.88, the inverse image of a stack F is given by the formula

(1) $$f^*(F) = Af^{-1}(LF)$$

where LF is the free split stack associated to F [4] p.39, where f^{-1} denotes the inverse image of LF as a category-object of the topos $\tilde{\underline{S}}$ and where A stands for "associated stack". Since there is a natural equivalence $F \longrightarrow LF$ and L is a morphism of bicategories we get a natural equivalence of split stacks $L(F^I) \longrightarrow (LF)^I$. Since the functor "inverse image of sheaves of sets" is left exact one gets a natural isomorphism $f^{-1}((LF)^I) \xrightarrow{\sim} (f^{-1}(LF))^I$ and it remains to find, for any prestack G over $\underline{S}'$ a natural equivalence $A(G^I) \longrightarrow (AG)^I$. One has a commutative square

$$\begin{array}{ccc} G & \xrightarrow{\ a\ } & AG \\ {\scriptstyle cG}\downarrow & & \downarrow{\scriptstyle cAG} \\ G^I & \xrightarrow[a^I]{} & (AG)^I \end{array}$$

where a is the structural map of AG . According to [4] p.77 it suffices to show that a^I is "bicouvrant" [4] p.72 , which is an easy consequence of the fact that a has this property. Q.E.D. .

Corollary 1.4. Let F and F' be left exact stacks on $\underline{S}$ and $\underline{S}'$, $m: F \longrightarrow f_*(F')$ be a morphism of stacks and $m': f^*(F) \longrightarrow F'$ the morphism associated to m by the universal property of the inverse image. Then m is left exact if and only if m' is .

This is a formal consequence of (1.3) .

§2.. Classifying topos of a stack.

Proposition 2.1. Let $\underline{S}$ be a left exact $\underline{U}$-site and C a prestack over $\underline{S}$ whose fibers are equivalent to categories which belong to $\underline{U}$ (C is said to be small) . Let us denote by J the coarsest topology on C such that the projection $p: C \longrightarrow \underline{S}$ is a comorphism [5] III 3.1 , and by $C\text{-}\underline{S}$ the category of sheaves on C for J with values in $\underline{U}$.

(1) J is defined by the pretopology whose covering families are those $(m_i : c_i \longrightarrow c), i \in I \in \underline{U}$, such that each m_i is $\underline{S}$-cartesian and such that $p(m_i), i \in I$,is a covering family.

(2) $C\text{-}\underline{S}$ is a $\underline{U}$-topos and the morphism $\pi: C\text{-}\underline{S} \longrightarrow \underline{S}$ defined by p is essential [i.e. π^* has a left adjoint $\pi_!$] . If C is left exact then $\pi_!$ is left exact.

(3) If $\underline{S}$ is a $\underline{U}$-topos and C is a stack, then the Yoneda functor $\varepsilon: C \longrightarrow C\text{-}\underline{S}$ is full and faithful and the composite $C \xrightarrow{\varepsilon} C\text{-}\underline{S} \xrightarrow{\pi_!} \underline{S}$ is equal to p .

Proof. (1) is an easy consequence of the definition of a comorphism and of the observation made in the proof of (1.1). Let $S_a, a \in A \in \underline{U}$, be a family of generators of $\underline{S}$ and $G_a, a \in A$, be a subset of $|C_{S_a}|$ which both belongs to $\underline{U}$ and contains an element of each isomorphism class of objects of the fiber C_{S_a} . The union of the G_a is a generator of the site (C,J) , hence this one is a $\underline{U}$-site and $C\text{-}\underline{S}$ is a $\underline{U}$-topos. Using (1) one sees easily that for any sheaf F on $\underline{S}$, Fp is a sheaf on C hence $\pi^*(F) = Fp$, hence π^* has a left adjoint hence π is essential. The last assertion of (2) follows from the fact that when C is left exact, p is the underlying functor of a morphism of sites $\underline{S} \longrightarrow C$. The first assertion of (3) follows readily from (1) and the patching condition for morphisms in C . For any $S \in |\underline{S}|$, and any $c \in |C_S|$ one has

$$\mathrm{Hom}(\pi_! \varepsilon(c), S) = \mathrm{Hom}(\varepsilon(c), \pi^*(S)) = \pi^*(S)(c) = \mathrm{Hom}(p(c), S)$$

by adjunction, Yoneda and the formula $\pi^*F = Fp$, and this concludes the proof.

2.2. Under the assumptions of (2.1) , C-$\underline{S}$ is called the <u>classifying topos of the</u> (pre)<u>stack</u> C . Note that a morphism of stacks $m: C \longrightarrow C'$ is a comorphism of sites and induces a morphism of topos $m\text{-}\underline{S}: C\text{-}\underline{S} \longrightarrow C'\text{-}\underline{S}$. If m is an equivalence, then so is $m\text{-}\underline{S}$. If C is a split stack one can define a split stack C^V whose fibers are the opposites of the fibers of C . Note that the underlying category of C^V <u>is not</u> the opposite C^o of C . Let us consider the category

$$(1) \qquad B_C(\underline{S}) = St_{\underline{S}}(C^V, SH(\underline{S}))$$

of morphisms of stacks $F: C^V \longrightarrow SH(\underline{S})$, where $SH(\underline{S})$ is the split stack whose fiber at $S \in |\underline{S}|$ is the category of sheaves on $\underline{S}/S$ [equivalent to $\underline{S}/S$ since $\underline{S}$ is a topos]. One has a natural functor

$$(2) \qquad \tau^*: \underline{S} \longrightarrow B_C(\underline{S}) \quad , \quad \tau^*(S)(c) = \varepsilon(S \times p(c)) \quad ,$$

where ε is the Yoneda functor of $\underline{S}/S$ . .

<u>Proposition 2.3</u>. If $\underline{S}$ is a $\underline{U}$-topos and C a split stack one has an equivalence of categories

$$(1) \qquad b: B_C(\underline{S}) \longrightarrow C\text{-}\underline{S} \quad , \quad b(F)(c) = F(c)(p(c))$$

and an isomorphism of functors $b.\tau^* \xrightarrow{\sim} \pi^*$.

2.3.1. Note that this proposition proves that $B_C(\underline{S})$ is a $\underline{U}$-topos equivalent to C-$\underline{S}$ even when C is not split since one can replace C by an equivalent split stack. Furthermore, by the universal property of the associated stack, $B_C(\underline{S})$ is equivalent to $B_{C'}(\underline{S})$ when C is the stack associated to some prestack C' .

Furthermore, Lawvere and Tierney have introduced for any category-object E of the topos $\underline{S}$, the topos of objects of $\underline{S}$ provided with operations of E . One can prove that this topos is equivalent to $B_C(\underline{S})$ where C is the split prestack defined by E hence also equivalent to C'-$\underline{S}$, where C' is the stack generated by C . Thus we have three constructions of the classifying topos.

2.3.2. For any split stack D , any map $f:T \longrightarrow S$ in $\underline{S}$ and any $s \varepsilon |D_S|$ we denote by s^f the inverse image of s by f and by $s_f : s^f \longrightarrow s$ the cartesian map given by the splitting. To define b completely one must define for any $m:c \longrightarrow c'$ in C an application $b(F)(m):b(F)(c') \longrightarrow b(F)(c)$. Let $f = p(m)$, $f:S' \longrightarrow S$. Since C is split there is a canonical factorisation $c' \xrightarrow{m'} c^f \xrightarrow{c_f} c$. Since F is cartesian one has a canonical isomorphism $i:F(c^f) \rightarrow F(c)^f$ which for the values at S' (or rather $id_{S'}$) of these sheaves induces a bijection $j:F(c^f)(S') \longrightarrow F(c)(f)$ and we take for $b(F)(m)$ the composite

$$F(c)(S) \xrightarrow{f(c)(\dot{f})} F(c)(f) \xrightarrow{j^{-1}} F(c^f)(S') \xrightarrow{f(m')(S')} F(c')(S') ,$$

where $\dot{f}:f \longrightarrow id_S$ is the terminal map in $\underline{S}/S$. It is easily checked that $b(F)$ is a functor, recalling that the underlying category of C^V is not the underlying category of C^o . The sheaf axiom for $b(F)$ is verified by using (2.1(1)): for a given family $(c_i \rightarrow c)$ it is a consequence of the fact that $F(c)$ is a sheaf and

F a cartesian functor. The functoriality with respect to F is obvious. To prove that b is an equivalence one constructs explicitly a functor

(2) $a:C\text{-}\underline{S} \longrightarrow B_C(\underline{S})$, $a(G)(c)(f) = G(c^f)$,

where $c\varepsilon|F|$ and $f:T \longrightarrow p(c)$ is a map in $\underline{S}$.

Proposition 2.4. Let $f:\underline{S}' \longrightarrow \underline{S}$ be a morphism of $\underline{U}$-topos and let C be a left exact stack over $\underline{S}$. One has an equivalence of categories

(1) $\mathrm{Top}_{\underline{S}}(\underline{S}',C\text{-}\underline{S}) \longrightarrow \mathrm{Stex}_{\underline{S}}(C,f_*SH(\underline{S}'))^o$, where the domain is the category of morphisms of $\underline{S}$-topos $n:\underline{S}' \longrightarrow C\text{-}\underline{S}$, where $f_*SH(\underline{S}')$ is the direct image by f of the stack of sheaves over $\underline{S}'$ [its fiber at $S\varepsilon|\underline{S}|$ is the category of sheaves over $\underline{S}'/f^*(S)$] and where the codomain is the opposite of the category of left exact morphisms of stacks $C \longrightarrow f_*SH(\underline{S}')$.

Since C is left exact and $\varepsilon:C \longrightarrow C\text{-}\underline{S}$ full and faithful, a morphism of topos $n:\underline{S}' \longrightarrow C\text{-}\underline{S}$ is nothing but a left exact functor $n^{-1}:C \longrightarrow \underline{S}'$, $n^{-1}=n^*\varepsilon$. Furthermore, since C is left exact there exists a cartesian section p^{-1} of C whose value at $S\varepsilon|\underline{S}|$ is the terminal object of the fiber C_S and p^{-1} is a morphism of sites defining $\pi:C\text{-}\underline{S} \longrightarrow \underline{S}$ since $\pi^*F = Fp$ for any sheaf F on $\underline{S}$. Hence an isomorphism of morphisms of topos $i:\pi n \xrightarrow{\sim} f$ is nothing but an isomorphism $i^{-1}:n^{-1}p^{-1} \xrightarrow{\sim} f^*$. In other words the category $\mathrm{Top}_{\underline{S}}(\underline{S}',C\text{-}\underline{S})^o$ is equivalent to the category M of pairs $(n^{-1}:C \longrightarrow \underline{S}', i^{-1}:n^{-1}p^{-1} \xrightarrow{\sim} f)$ where n^{-1} is continuous and left exact. Let $Ar(\underline{S}')$ be the category whose objects are arrows of $\underline{S}'$ and let $b:Ar(\underline{S}') \longrightarrow \underline{S}'$, $b(X \longrightarrow Y) = Y$. Since every object $c\varepsilon|C|$ determines a terminal map $c \longrightarrow p^{-1}(p(c))$, a pair (n^{-1},i^{-1}) can be viewed as a functor $n':C \longrightarrow Ar(\underline{S}')$ such that

$bn' = f^*p$ and which is left exact [the continuity condition disappears by (2.1 (1))]. Since b makes a stack over $\underline{S}'$ out of the category $Ar(\underline{S}')$, by the very definition of the direct image of a stack, n' is nothing but a functor $n'':C \longrightarrow f_*Ar(\underline{S}')$ and, since n' is left exact, n'' is $\underline{S}$-cartesian and left exact, hence an object of $Stex_{\underline{S}}(C,Ar(\underline{S}'))$. The conclusion follows since $Ar(\underline{S}')$ is equivalent to $SH(\underline{S}')$.

According to the proof, the morphism of topos $n:\underline{S}' \longrightarrow C\text{-}\underline{S}$ which corresponds to a left exact morphism of stacks $n'':C \longrightarrow f_*Ar(\underline{S}')$ is characterized up to unique isomorphism by the equality $n^*\varepsilon = dqn''$

(2) $$C \xrightarrow{n''} f_*Ar(\underline{S}') \xrightarrow{q} Ar(\underline{S}') \xrightarrow{d} \underline{S}' \ ,$$

where q is the first projection of $f_*Ar(\underline{S}') = Ar(\underline{S}') \times_{\underline{S}'} \underline{S}$, d the "domain functor" and ε the Yoneda functor.

Corollary 2.5. If C is left exact one has an equivalence (1)

(1) $$Top_{\underline{S}}(\underline{S}',C\text{-}\underline{S}) \longrightarrow Stex_{\underline{S}'}(f^*(C),SH(\underline{S}'))^{o} \ .$$

This follows immediately from (2.4),(1.4) and the universal property of the inverse image $f^*(C)$ of C. This gives the universal property of $C\text{-}\underline{S}$ in the bicategory of $\underline{S}$-topos.

Corollary 2.6. Let $C' = f^*(C)$. One has a commutative square of morphisms of topos

(1) $$\begin{array}{ccc} C\text{-}\underline{S} & \xleftarrow{C\text{-}f} & C'\text{-}\underline{S}' \\ \downarrow & & \downarrow \\ \underline{S} & \xleftarrow[f]{} & \underline{S}' \end{array}$$

which is bicartesian.

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(1) $Stex_{\underline{S}}(\ ,\ )$ stands for "category of left exact morphisms of stacks".

This means that for any morphism of topos $g:\underline{S}''\longrightarrow \underline{S}'$ the functor given by composition with C-f

(2) $$\mathrm{Top}_{\underline{S}'}(\underline{S}'', C'\text{-}\underline{S}') \longrightarrow \mathrm{Top}_{\underline{S}}(\underline{S}'', C\text{-}\underline{S})$$

is an equivalence. By the very definition of C' [4] p.87, one has a commutative square

(3) $$\begin{array}{ccc} C & \xrightarrow{\phi\text{-}1} & C' \\ p\downarrow & & \downarrow p' \\ \underline{S} & \xrightarrow{f^*} & \underline{S}' \end{array}$$

where ϕ^{-1} is cartesian. Furthermore ϕ^{-1} is left exact by (1.3). By (1.4) and the univsal property of $C' = f^*(C)$, for any $g:\underline{S}''\longrightarrow \underline{S}'$, the functor

(4) $\mathrm{Stex}_{S'}(C', g_*SH(\underline{S}'')) \longrightarrow \mathrm{Stex}_S(C, f_*g_*SH(\underline{S}''))$, $u \longrightarrow u\phi^{-1}$, is

an equivalence. By (2.4) the proof is now an exercise about universal properties in bicategories.

§3. Generating stack of a U-topos.

The question of defining a relative notion of generators has been raised by Lawvere and Tierney. We propose here an answer in the language of U-topos. It is clear that Prop.(3.3) is still valid when working in their framework and that (3.2) is not.

Definition 3.1. Let $f:\underline{X}\longrightarrow \underline{S}$ be a morphism of U-topos. A generating stack of f is a full substack C of $F = f_*(\mathrm{Ar}(\underline{X}))$ which is small (2.1) and such that, for any $S\varepsilon|\underline{S}|$ and any $x\varepsilon|F_S|$, there exists a covering family $(S_i \longrightarrow S)$, $i\varepsilon I$, in $\underline{S}$ and for each $i\varepsilon I$ a covering family $(c_{i,j} \longrightarrow x_i)$ in the fiber $F_S = \underline{X}/f^*(S)$, with $c_{i,j}\varepsilon|C|$, where x_i is the inverse image of x by $S_i\longrightarrow S$. A generating stack C is said to be left exact if C and the inclusion functor $i: C \longrightarrow F$ are left exact.

Let us recall that the category of arrows of $\underline{X}$ provided with the codomain functor $Ar(\underline{X}) \longrightarrow \underline{X}$ is a stack. Hence its direct image F is a stack whose fiber at $S \varepsilon |\underline{S}|$ is the topos $\underline{X}/f^*(S)$ and the inverse image functor $F_u: F_S \longrightarrow F_{S'}$ associated to a map $u: S' \longrightarrow S$ in $\underline{S}$ is nothing but pull-back along $f^*(u): f^*(S') \longrightarrow f^*(S)$. Hence F is a left exact stack and the condition that a full substack C of F is left exact is that each fiber C_S is stable by finite limits in the fiber F_S .

Proposition 3.2. Any $\underline{S}$-topos admits a left exact generating stack.

Let us choose a generator $(S_i), i \varepsilon I \varepsilon \underline{U}$, of $\underline{S}$ and for each $i \varepsilon I$ a full subcategory C_i of F_{S_i} stable by finite limits, generating F_{S_i} and equivalent to a category which belongs to $\underline{U}$. Let us define C as the full subcategory of F whose objects of projection $S \varepsilon |\underline{S}|$ are those $x \varepsilon |F_S|$ such that there exists a covering family $(c_a: S_a \longrightarrow S)$, such that each S_a is one of the S_i and the inverse image of x by c_a is isomorphic to an object of C_i . This condition being local on $\underline{S}$, it is clear that C is a full substack of F and even a left exact one since F is left exact. Furthermore C is small since for each $S \varepsilon |\underline{S}|$ the set of classes of equivalent covering families $(S_a \longrightarrow S)$ as above belongs to $\underline{U}$. Eventually C is a generating stack since any $S \varepsilon |\underline{S}|$ can be covered by the S_i .

Proposition 3.3. Let $\underline{S}$ be a $\underline{U}$-topos and C a generating stack of an $\underline{S}$-topos $f: \underline{X} \longrightarrow \underline{S}$. Then C-$\underline{S}$ is an $\underline{S}$-topos and there exists an $\underline{S}$-morphism of topos $n: \underline{X} \longrightarrow C$-$\underline{S}$ such that $n_*: \underline{X} \longrightarrow C$-$\underline{S}$ is full and faithful [in other words $\underline{X}$ is a subtopos of C-$\underline{S}$] .

3.3.1. We note first that since C is small, $C\text{-}\underline{S}$ is a $\underline{U}$-topos . Furthermore there exists a left exact generating stack C' of $\underline{X}$ containing C and such that each object of C' is a finite limit of objects of C . Hence the inclusion $C \longrightarrow C'$ induces an equivalence between the $\underline{S}$-topos $C\text{-}\underline{S}$ and $C'\text{-}\underline{S}$ and this fact allows us to assume that C is left exact. Since the inclusion $i:C \longrightarrow F$, $F = f_*\mathrm{Ar}(\underline{X})$, is left exact one has an $\underline{S}$-morphism $n:\underline{X} \longrightarrow C\text{-}\underline{S}$, (2.4) , whose inverse image functor $n^*:C\text{-}\underline{S} \longrightarrow \underline{X}$ is such that its composition with the Yoneda functor $\varepsilon:C \longrightarrow C\text{-}\underline{S}$ is equal to the composite of

(1) $$C \xrightarrow{i} F \xrightarrow{q} \mathrm{Ar}(\underline{X}) \xrightarrow{d} X \quad , \quad (2.4(2)) .$$

For any $c\varepsilon|C|$ and any $X\varepsilon|\underline{X}|$ one has $n_*(X)(c) = \mathrm{Hom}(\varepsilon(c),n_*(X)) = \mathrm{Hom}(n^*\varepsilon(c),X) = \mathrm{Hom}(dqi(c),X) = \mathrm{Hom}_S(i(c),X\times f^*(S))$ where the last set of morphisms is taken in the fiber $\underline{X}/f^*(S)$ of F with $S = p(c)$, and the last equality sign is justified by the definition of F as a fibered product. Hence the formula

(2) $$n_*:\underline{X} \longrightarrow C\text{-}\underline{S} \quad , \; n_*(X)(c) = \mathrm{Hom}_S(i(c),X\times f^*(S)) \quad , \; S = p(c) .$$

3.3.2. To prove that n_* is full and faithful we will first compose it with the inverse $a:C\text{-}\underline{S} \longrightarrow B_C(\underline{S})$ of (2.3(1))

(3) $$an_*:\underline{X} \longrightarrow B_C(\underline{S}) \quad , \; an_*(X)(c) = \underline{\mathrm{Hom}}_S(i(c),X\times f^*(S)) \; , \; S = p(c) \; , \quad c\varepsilon|C| \; ,$$

the above formula being justified by (2.3(2)) , where $\underline{\mathrm{Hom}}_S(u,v)$ stands for the sheaf (over S) of S-morphisms between the objects u and v of the fiber at S of the stack F . Let us prove that (3) is the effect on the fibers at the terminal object of $\underline{S}$ of a morphism of stacks

(4) $m:F \longrightarrow ST(C^V, SH(\underline{S}))$,

where $ST(A,B)$ stands for the (split) stack of morphisms of stacks between A and B [internal Hom in the bicategory of stacks [4] p.57, 77] , whose fiber at $S\epsilon|\underline{S}|$ is the category of morphisms $A/S \longrightarrow B/S$ of stacks over $\underline{S}/S$. We obtain (4) by composition of

(5) $F \xrightarrow{y} ST(F^V, SH(\underline{S})) \xrightarrow{j} ST(C^V, SH(\underline{S}))$

where j is induced by composition with $i:C \longrightarrow F$ and where y is a "relative Yoneda functor" defined by

(6) $y(a)(b) = \underline{Hom}_S(b, a^f)$,

where $f:T \longrightarrow S$ is a map in $\underline{S}$ and $a\epsilon|F_S|$, $b\epsilon|F_T|$. One should note that the restriction of y to the terminal fiber of F is also the restriction of the composite $F \xrightarrow{\varepsilon} F\text{-}\underline{S} \xrightarrow{a} B_F(\underline{S})$, (2.1(3)), (2.3(2)). By localisation it follows that the restriciton of y to each fiber is full and faithful hence y is such. On the other hand, since any object of F can be covered for the canonical topology of F by objects of $i(C)$ and since i is full and faithful it is easy to show that j is also full and faithful and the conclusion follows.

Proposition 3.4. Fibered products exist in the bicategory of $\underline{U}$-topos.

According to (3.2) and (3.3) any morphism of $\underline{U}$-topos $\underline{X} \longrightarrow \underline{S}$ can be factored in $\underline{X} \xrightarrow{n} C\text{-}\underline{S} \xrightarrow{\pi} \underline{S}$ where n_* is full and faithful and where C is a left exact small stack over $\underline{S}$. By (2.6) the pullback of π along any morphism of $\underline{U}$-topos $f:\underline{S}' \longrightarrow \underline{S}$ exists. On the other hand the pull-back of n along any morphism of $\underline{U}$-topos $y:\underline{Y} \longrightarrow C\text{-}\underline{S}$ exists because $\underline{X}$ is a subtopos of $C\text{-}\underline{S}$ hence is defined by some topology J on $C\text{-}\underline{S}$ and it suffices to take as a pullback the subtopos of $\underline{Y}$ defined by the finest topology

J' on $\underline{Y}$ such that the inverse image functor $y^*:C\text{-}\underline{S} \longrightarrow \underline{Y}$ is continuous. The conclusion follows by transitivity of pullback in a bicategory.

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