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Dear Lipman,

Following your letter I am sending you the outline of a program of work for the local Ricard scheme, which was essentially what I had in mind in SGA 2. It would be very nice indeed if you could accomplish this program, or part of it. I would be interested to know about any progress you make, or about any questions that will arise, in this connection.

Sincerely yours

A Grothendieck

Local Picard schemes.

henselian enough? dim ≥ 2, assume domain }

Let A be a complete local ring, ~~Mx~~ $S = \mathrm{Spec}(A)$, $U = S - \{s\}$ the complement of the closed point of S. We wish to interprete the Picard ~~gxx~~ group $\mathrm{Pic}(U)$ as ~~some~~ the group $Q(k)$ of k-valued points of a group-scheme Q defined over the residue field k of A. Assume for simplicity that U is regular, and that there exists a resolution of singularities for S not changing U, i.e. a proper morphism

$$f : X \longrightarrow S$$

inducing an isomorphism $f^{-1}(U) \xrightarrow{\sim} U$, such that X be regular and equal to the closure of $f^{-1}(U)$ (this open set will be identified with U).

Just suppose f bir'l.

Let X_o be the special fiber of X, so that

$$U = X - X_o \ .$$

Because X is regular, we have an exact sequence

(1) $$D \longrightarrow \mathrm{Pic}(X) \longrightarrow \mathrm{Pic}(U) \longrightarrow 0 \ ,$$

where D is the group of divisors on X concentrated on X_o (which is a free group of finite type $\underline{Z}^I$, I being the set of irreducible components of X_o). On the other hand, by the existence theorem for coherent sheaves, it is known that

Seems that one assumes here that $m\mathcal{O}_X$ is invertible!

(2) $$\mathrm{Pic}(X) \simeq \varprojlim_n \mathrm{Pic}(X_n) \ ,$$

where $X_n = X \times_S S_n$, $S_n = \mathrm{Spec}(A/\underline{m}^{n+1})$. Assume we can find group schemes P_n over k and isomorphisms

(3) $$\mathrm{Pic}(X_n) \xrightarrow{\sim} P_n(k) \ ,$$

and transition morphisms $P_m \rightarrow P_n$ ($m \geq n$) compatible with the transition morphisms for the $\mathrm{Pic}(X_n)$. Denoting by P the pro-group (P_n), or if one prefers, in case the transition morphisms $P_m \longrightarrow P_n$ are affine, the inverse limit (EGA IV 8) of the P_n, we get a pro-group scheme resp. a group-scheme P, and an isomorphism

(4) $$\mathrm{Pic}(X) \xrightarrow{\sim} P(k) .$$

The map $D \to \mathrm{Pic}(X) \simeq P(k)$ défines a morphism of (pro) groups

(5) $$D_k \to P \quad ,$$

where D_k is the constant group-scheme with value D. Assuming that the cokernel of (5) exists in a reasonable sense a a pro-group scheme or a group-scheme, let Q be this cokernel. We get, in virtue of (1), an injective homomorphism

(6) $$\mathrm{Pic}(U) \hookrightarrow Q(k) \quad ,$$

the obstruction for an element of $Q(k)$ to be in the image being in $H^1(k, D'_k)$, where D'_k is the image of D_k in P (it is again a constant group, with value a quotient D' of D). If k is separably closed, or if D' is torsion-free, for example is D' is equal to D i.e. (5) is injective, then this obstruction vanishes, and (6) is an isomorphism. Thus Q looks a good candidate for a "local Picard scheme".

To construct the P_n's , assume first that A is a k-algebra, i.e. that a lifting of the residue field k exists, and has been choosen. Then the natural candidate for P_n (the only one I could think of, indeed !) would be

(7) $$P_n = \underline{\mathrm{Pic}}_{X_n/k} \quad .$$

We know from Murre that the right had side is indeed representable by a scheme locally of finite type over k. There is however the usual trouble that we have inded a canonical injective morphism

(8) $$\mathrm{Pic}(X_n) \hookrightarrow P_n(k) \quad ,$$

but in general we are not sure it is bijective. However it is if k is separably closed; more specifically, the obstruction for an element of $P_n(k)$ to be in the image of $\mathrm{Pic}(X_n)$ lies in the Brauer group of the Artin ring $B_n = H^0(X_n, \underline{O}_{X_n})$. Assuming U hence X hence X_n to be connected, this Artin ring is local, and it's Brauer group coincides with the Brauer group of its residue field k_n. Of course, ~~thus~~ for an element of $P(k) = \lim P_n(k)$, these various obstructions match together and come from an element of the Brauer group of the residue field $k' \simeq \bigcap_n k_n$ of the ring $H^0(X, \underline{O}_X) = \varprojlim_n H^0(X_n, \underline{O}_{X_n})$, ~~of~~ which is just the normalisation of A. If for instance A is normal, then $k' = k$ and the obstruction lies in $Br(k)$.

NB In view of the assumption U regular hence normal, normality of A is not a real restriction (otherwise replace A by normalization!)

Whatever this be, defining Q as before, we still do get an injective homomorphism (6), but even if (5) is injective, we cannot be sure that (6) is bijective. We are however if k is separable closed. Therefore, we still ~~get~~ can more or less describe Q (or at least Q(k)) in terms of local Picard groups, in terms of Galois descent from the completion A' of the strict henselization of A: if $U' = S' - \{s'\}$, where $S' = \mathrm{Spec}(A')$, and if

$$\pi = \mathrm{Gal}(\bar{k}/k) , \tag{9}$$

we will have an isomorphism

$$Q(k) \simeq \underline{\mathrm{Pic}}(U')^{\pi} , \tag{10}$$

~~Thisxisxperf provided Q' is torsion free, for instance (5) injective.~~

This is perfectly natural and in accordance with the familiar phenomena when dealing with global Picard schemes. Therefore Q looks like the natural object to be called a ~~Picard~~ local Picard scheme.

To make sure it makes a sense, one will have however to analyze somewhat the inverse system (P_n), and the morphism (5). Moreover, to feel really secure, one should check that up to canonical isomorphism, the group scheme Q together with (10) does not depend on the choice of

the resolution; this should not be hard whenever we have existence of resolutions in the strong sense of Hironaka, for instance in char. 0
so that two resolutions can be dominated by a third one.
of if dim A = 2, (I should point out that, in any case, the algebraic structure thus defined on Pic(U) depends in an essential way on the choice of the lifting of the residue field; an instructive example on which to study the dependance of this structure on the lifting is the one where A is the completion of the local ring at the origin of the projecting cone of an elliptic curve (here Q is isomorphic to the elliptic curve in question, for the <u>given</u> lifting of k ...).

I have to correct an inaccuracy that has slipped in, when defining Q in the case when k is not separably closed, so as to get (10). One will then have to replace the constant group scheme D_k by the twisted constant group scheme $\underline{D}$, deduced by descent from the constant group-scheme occuring after passage from k to the separable closure. ~~In terms of the components X_o^i of X_o, this can be defined as the direct image, under the morphism $Spec(B_{o\ sep})$ Spec(k), of the constant sheaf $\underline{Z}$ on the spectrum of the largest separable subalgebra $B_{o\ sep}$ of B_o~~ This can be made explicit, in terms of the separable algebraic closures k_i of k in the function fields of the irreducible components X_o^i of X_o , as the direct image of the constant sheaf $\underline{Z}$ under the projection $Spec(\prod k_i) \rightarrow Spec(k)$

To study the inverse system (P_n) , use the fact (SGA 6 XII) that the transition morphisms are affine, which gives a sense to $P = \varprojlim P_n$ as an actual group-scheme. It's Lie algebra is the inverse limit of the Lie algebras of the P_n , which are $H^1(X_n, \underline{O}_{X_n})$, therefore we get

(11) $$\underline{\mathrm{Lie}}(P) \simeq H^1(X, \underline{O}_X) \quad .$$

As $R^1 f_*(\underline{O}_X)$ is coherent and concentrated at the origin, it follows that its group of sections $H^1(X, \underline{O}_X)$ is of finite dimension. In fact, it seems, with a little care, that replacing if necessary X by a suitable model dominating it, we may assume that there exists an invertible ideal J on X, defining a subscheme Z of X having same support as X_o, and such that $\underline{L} = J/J^2$ be an ample sheaf on Z. This will imply that, replacing the system $(X_n)_n$ by the system $(Z_n = V(J^{n+1}))_n$, which defines an isomorphic pro-object of Picard -schemes, we obtain an inverse system of group-schemes $P'_n = \underline{\mathrm{Pic}}_{Z_n/k}$, where for large n the transition morphisms $P'_{n+1} \to P'_n$ are isomorphisms (because $H^1(Z_o, L^{\otimes n}) = H^2(Z_o, \underline{L}^{\otimes n}) = 0$). Thus it seems that, under the assumptions made from the start, one can prove that the pro-group (P_n) is essentially constant, hence its inverse limit P is in fact a group scheme locally of finite type over k . This is certainly so, at least, if dim A = 2, A normal, as follows from the negative definiteness of the intersection matrix of the components of X_o .

This P seems to be definable in quite a reasonable way, and one still has to investigate the morphism (5) (or more accurately, the morphisme $\underline{D} \to P$ defined by descent from the case where k is separably closed; but then we may as well assume k separably closed from the start (which we will do). We would be particularly happy if this were a closed immersion, or what amounts to the same, if the morphism

(12) $$D \longrightarrow NS(X_o)$$

is injective, were $NS(X_o) = \underline{NS}_{X_o/k}(k)$, $\underline{NS} = \underline{\mathrm{Pic}} / \underline{\mathrm{Pic}}^o$. (The equivalence comes from the fact that $P = \underline{\mathrm{Pic}}_{Z_n/k}$ for some large n, and that, etc as recalled above, the morphism $\underline{\mathrm{Pic}}_{Z_n/k} \to \underline{\mathrm{Pic}}_{X_o/k}$ is affine, and hence $NS(Z_n) \to NS(X_o)$ has finite kernel.) I expect that this is always the case, In case A is normal of dimension 2, this follows from Du-Val - Mumford's negative definiteness. It would be nice to have a proof in general(perhaps through a negative definiteness of an intersection matric $X^i.C^j$, where for every irreducible component X^i of X_o , C^i denotes a suitable curve on X^i , for instance defined in terms of a suitable ~~ample~~ relatively ample sheaf on X by taking intersections of corresponding hypersurface sections on the X^i 's ?).

In case (12) is indeed injective, we have no trouble defining

(13) $$Q = P/\underline{D}$$

as a group-scheme locally of finite type over k. We then have

(14) $$P^o \xrightarrow{\sim} Q^o \quad ,$$

an isomorphism for the connected components, and

(15) $$Q/Q^o = (P/P^o)/\underline{D} \simeq \underline{NS}_{Z_n/k} / \underline{D} \quad ,$$

where n is chosen large enough. Of course (14) and (11) yield an isomorphism

(16) $$\underline{\mathrm{Lie}}(Q) \simeq H^1(X,\underline{O}_X) \quad .$$

Let us remark by the way that we have an injective restriction morphism

(17) $$H^1(X,\underline{O}_X) \hookrightarrow H^1(U,\underline{O}_U) = H^2_s(S)$$

as one sees by remarking that, with the above notations for $\underline{J}$, $\underline{L}$, the sheaf ~~of~~ $i_*(\underline{O}_U)/\underline{O}_X$ ($i: U \to X$ the inclusion) has a composition series whose quotients are the $\underline{L}^{-n}$, $n \geq 1$, whose H^o is zero, and the map (17) is just the canonical map $H^1(X,\underline{O}_X) \to H^1(X,i_*(\underline{O}_U))$. Thus, the Lie algebra of the local Picard scheme is identified with a subspace of $H^1(U,\underline{O}_U)$. (This suggests that if A is Cohen-Macaulay of dim. ≥ 3, Q is discrete, hence Pic(U) countable etc).

The previous constructions rely heavily on the existence and choice of a field of representatives. Even this being granted, the question of caracterising Q as an actual scheme over k, i.e. as a functor on the category of k-algebras, has not been touched. In fact, the description I can give of this functor is (tautological, and is) in terms of a particular choice of a resolution X, and it really gives a description of P rather than Q . One then still has to define Q as the cokernel of the silly morphism $\underline{D} \rightarrow P$. As a matter of fact, even if we restrict to arguments k' which are, say, finite extensions of k (but not necessarily separable ones), I cannot describe Q(k') in terms of local Picard groups of local rings as $A' = A \otimes_k k'$. The trouble, of course, is that $X \otimes_k k'$ will in general no longer be a resolution of S', i.e. it will no longer be regular. However, such geometric descriptions will be possible provided we take as arguments algebras which are geometrically regular over k, so that tensoring with them will not destroy the regularity of X. (Thus, the case when k' is a power series ring, for instance the completion of the local ring of Q at the origin, Q being smooth, is of interest ...) Let's now try to be specific. By definition more or less, for any affine scheme T over k, we have

affineness of T used to have $H^1(T,\underline{O}_T) = H^2(T,\underline{O}_T) = 0$

$$(18) \qquad P(T) = \varprojlim_n P_n(T) \hookleftarrow \varprojlim_n \mathrm{Pic}(X_{n_T}) \quad \mathrm{Pic}(T) \qquad (X_{n\,T} = X_n \times_k T\ ) .$$

In fact, it is not hard to check that the left hand side functor is the étale sheaf (on $(Sch)_{/k}$) associated to the right-hand side functor. This comes from the more precise statement that the obstruction, for an element of P(T), to come from an element of $\varprojlim \mathrm{Pic}(X_{nT})$, lies in $H^2(T_{ét}, \underline{G}_m)$ (at least for T affine, (which is used to insure that it's H^1 and H^3 with values in $\underline{O}_T$ vanish); for simplicity, I have assumed here that A is normal). As we have seen on the other hand that P is locally of finite presentation over k, it commutes (as a functor) to filtering direct limits of rings, and therefore it is known when we know it's restriction to arguments which are of finite type over k, and a fortiori ~~xxx~~ noetherian. For such (noetherian) arguments, using the existence theorem for coherent sheaves, we can give a more geometric expression for the right hand side of (18), ~~namely~~ Assume for simplicity $T = \mathrm{Spec}(B)$ affine, and let

$$C = A\hat{\otimes}_k B = \lim A_n \otimes_k B \quad ,$$

which is an adic noetherian ring augmented to B (the augmentation ideal being an ideal of definition). Consider ~~$X\hat{\otimes}_k B$~~ $X\otimes_A C$, which is a proper scheme over C. We then have $X\hat{\otimes}_k B \overset{\text{dfn}}{=}$

(19) $$\mathrm{Pic}(X\hat{\otimes}_k B) \xrightarrow{\sim} \varprojlim_n \mathrm{Pic}(X_{nB}), \quad \mathrm{Pic}(X\hat{\otimes}_k B)/\mathrm{Pic}(B) \subset P(B)$$

and thus on noetherian arguments B, P appears as the étale sheaf associated to the functor $B \mapsto \mathrm{Pic}(X\hat{\otimes}_k B)/\mathrm{Pic}(B)$, (which is a separated presheaf,) this statement being complemented by the description given above of the group for obstructions for an element of P(B) to belong to $\mathrm{Pic}(X\hat{\otimes}_k B)$.

As for ~~that~~ $B \mapsto Q(B)$, this appears as the ~~xxxx~~ étale sheaf associated with the functor

(20) $$B \longmapsto \mathrm{Pic}(X\hat{\otimes}_k B)/(\mathrm{Pic}(B) + \underline{D}(B)) \; (\hookrightarrow Q(B))$$

where $\underline{D}$ is the twisted constant group over k defined above. For an element of Q(B) to belong to the first hand side there are two successive obstructions, the first is in $H^1(\mathrm{Spec}(B),\underline{D})$, which vanishes for instance if $\underline{D}$ is constant (for instance k separably closed) and B normal, the second is in the Brauer group of B, and vanishes for instance if X_0 has a zero-cycle of degree 1.

Assume for simplicity that the irreducible components of X_0 are geometrically irreducible, i.e. $\underline{D}$ is constant, and assume B geometrically regular over k. This second condition implies that $X\hat{\otimes}_k B$ is regular, and denoting by $U\hat{\otimes}_k B$ the inverse image of U in this scheme, ~~the first condition~~ this together with the first condition implies that ~~the left hand side of (20)~~ $\mathrm{Pic}(X\hat{\otimes}_k B)/\underline{D}(B)$ is isomorphic to $\mathrm{Pic}(U\hat{\otimes}_k B)$. This shows that on the sub-category of arguments B which are noetherian and geometrically regular, Q is the étale sheaf associated to the presheaf

$$(21) \qquad B \longmapsto \mathrm{Pic}(U\hat{\otimes}_k B)/\mathrm{Pic}(B) \quad (\hookrightarrow Q(B)) \quad .$$

The obstruction for an element of Q(B) to belong to $\mathrm{Pic}(U\hat{\otimes}_k B)/\mathrm{Pic}(B)$ is in the Brauer group Br(B). Thus, under very stringent conditions on B at least (being geometrically regular), we get a description of Q(B) in terms of actual local Picard groups. I should have stated that $U\hat{\otimes}_k B$ can also be interpreted as the inverse image of U in the scheme $\mathrm{Spec}(A\hat{\otimes}_k B)$, and thus makes a sense independently of the choice of a particular resolution.

One may wish to construct a local Picard scheme ~~without~~ also in case A is not equi-caracteristic, which implies that car.k > 0. It seems likely that this can be done if k is perfect. The key-point here ~~seems to~~ is the construction of schemes P_n , which leads us to the following

Problem Let A be a local Artin ring with perfect residue field k of car. $p > 0$, X a proper scheme over A. Give a "natural" construction of a group scheme locally of finite type P over k, together with an imbedding

$$(22) \qquad \mathrm{Pic}(X) \hookrightarrow P(k) \ ,$$

the obstruction for an element of P(k) to belong to Pic(X) (being in Br $H^0(X,\underline{O}_X)$, and hence) vanishing if k is algebraically closed.

Here is a candidate for a functor, which may turn out to be representable, in which case this would be P. Let $\underline{A}$ be the ring-scheme over k defined by A (see Greenberg's papers, or Serre's in Bulletin Soc. Math.), so that

(23) $$A \simeq \underline{A}(k) \ .$$

For every algebra B over k, consider the ring $\underline{A}(B)$, which is an algebra over $\underline{A}(k) = A$. We may thus consider

$$X_B \overset{\text{dfn}}{=} X \otimes_A \underline{A}(B) \ ,$$

and consider the functor

(24) $$B \longmapsto \mathrm{Pic}(X_B) \qquad .$$

This of course may not be representable, even if A=k . However, it may turn out that the fppf sheaf associated to this functor is representable, and this then would give the looked for candidate for P. In any case, Artin's work gives us a very handy set of necessary and sufficient conditions for a group functor to be representable by a scheme locally of finite type, and in principle it should be possible to decide wether or not the sheaf just considered satisfies to these conditions or not. If so, one may expect that the developments on local Picard schemes given for the case when there is a field of representatives will carry over to the case when the residue field is perfect with car. $>$ 0.

Could perhaps consider a functor over R, a discrete val'n ring, where $R \to A$ is a ring of representatives, and try to represent it (in some sense) by an algebraic group over R.

In the construction of a local Picard scheme Q, we have seemed to use in a technically essential way the assumption that U is regular (plus resolution). I would conjecture that a reasonable local Picard scheme Q, locally of finite type over k, can be constructed also without this assumption (provided that a field of representatives is given, or the residue field is perfect of char. $p > 0$, of course). Instead of a resolution of singularities, one may then wish to find a proper surjective morphism

$$f : X \longrightarrow S$$

inducing an isomorphism $f^{-1}(U) \xrightarrow{\sim} U$, and such that the corresponding map

$$\mathrm{Pic}(X) \longrightarrow \mathrm{Pic}(U)$$

is <u>surjective</u>. We then could repeat the construction of P, as $\lim \underline{\mathrm{Pic}}_{X_n/k}$, as above, and of Q as $\mathrm{Coker}(\underline{D} \to P)$, where $\underline{D}$ is a group scheme corresponding to divisors concentrated on X_0 (possibly no longer discrete ?). Another way of approach would be to take a resolution of singularities $f : X \to S$, but where now we cannot assume any longer that $f^{-1}(U) \to U$ is an isomorphism, and to describe a scheme P in terms of invertible sheaves on X, together with <u>descent data</u> of $\underline{L}|f^{-1}(U)$ to U. A nice test for the yoga of local Picard schemes without regularity assumptions on U would be to see if it is true in general that for A Cohen-Macaulay of dim ≥ 3 (more generally, for $H^2_{\underline{m}}(A) = 0$), the group Pic(U) is indeed countable, and for separably closed residue field is invariant under <u>separable</u> field extension $k \to k'$, A being replaced by $A \hat{\otimes}_k k'$ (which would express the expected discrete structure of the local Picard scheme).