Massy 22.8.1969

Dear Lipman,

Thanks for your letter. The fact that for a proper scheme X over a field k, every formal arc in $\underline{\mathrm{Pic}}_{X/k}$ through <u>the origin</u> defines an element of $\mathrm{Pic}(X \otimes_k k[[t]])$, and a fortiori in $\mathrm{Pic}(X \otimes_k k((t)))$, comes from the fact that the obstruction to finding the element in $\mathrm{Pic}(X \otimes_k k[[t]])$ is in the Brauer group $\mathrm{Br}(k[[t]])$, and that the reduction map

$$\mathrm{Br}\, k[[t]] \longrightarrow \mathrm{Br}\, k$$

is <u>injective</u>.

The result

$$A \text{ rational} \iff \text{local } \underline{\mathrm{Pic}} \text{ 0-dimensional}$$

is true whenever A is given with a field of representatives $k \xrightarrow{\sim} k(A)$, perfect or not. The reason for this is that, if X is a desingularization, the explicit construction of local $\underline{\mathrm{Pic}}$ yields that it's tangent space to the origin is can. isom. to $H^1(X, \mathcal{O}_X)$. This does not contradict your example, as formation of the local <u>Pic</u> commutes with <u>separable</u> field extensions $k \to k'$ only (just as taking desingularizations).

The fact that if R is a complete local ring of dim ≥ 3 which is CM (more generally, such that $H^2_m(R) = 0$), with field of representation $k \subset R$, $k \xrightarrow{\sim} k(R)$, should have discrete <u>Pic</u> should follow from resolution (which allows to <u>construct</u> <u>Pic</u>) and from the explicit construction, as this yields a canonical embedding of the tangent space to the local <u>Pic</u> into $H^2_m(R)$. As a consequence, we get that $H^2_m(R) = 0$ $\Longrightarrow$ Pic(Spec R − $\{m\}$) countable. I forgot to say that Spec R is supposed to have isolated singularity (but conjecturally one should be able to construct the local Picard scheme without this assumption). For the case of unequal characteristics, I would still conjecture that $H^2_m(R) = 0 \Longrightarrow$ Pic(Spec R − $\{m\}$) countable. It may follow from the same argument, for <u>perfect</u> residue field, if we succeed to construct still the local <u>Pic</u> scheme. Maybe you should try this! Artin's general representability theorem should allow us just to check if it is OK.

or not. If you are interested, I can tell you which functor one should represent.

Sincerely yours

A Grothendieck